Quadratic Equations and Functions

1. Introduction

A quadratic equation in one unknown is an equation of the form $ax^2+bx+c=0$, where $a\neq 0$. When we solve a linear equation, we may transpose the terms and leave the unknown on one side of the equation. However, this is often not the case for a quadratic equation. There are other methods to solve a quadratic equation, e.g. by factorization, by completing the square and by the quadratic formula. Furthermore, for a linear equation in the form of $mx+n=0$, where $m\neq 0$, there is always a solution $x=-\frac{n}{m}$, which is a real number. On the contrary, a quadratic equation may have two real roots, one double root or no real roots.

2. Solving quadratic equations

There are several methods to solve a quadratic equation. Some quadratic expressions can be factorized and then the equation is easy to solve.

Theorem 2.1.

If $pq = 0$, then $p = 0$ or $q = 0$.

From theorem 2.1, if a quadratic equation $ax^2+bx+c=0$ can be factorized to the form $(px+q)(rx+s)=0$, then we have $px + q = 0$ or $rx + s = 0$, which gives $x=-\frac{q}{p}$ or $x=-\frac{s}{r}$ respectively.

Example 2.1.

Solve $x^2+5x+6=0$.

Solution.

We try to factorize x^2+5x+6. This can be written as $(x+3)(x+2)$.

Thus we have

$$\begin{aligned} x^2+5x+6&=0 \\ (x+3)(x+2)&=0 \\ x+3=0 \quad &\text{or} \quad x+2=0 \\ x=-3 \quad &\text{or} \quad x=-2. \end{aligned}$$

Example 2.2.

Solve $6x^2+13x+6=0$.

Solution.

$$
\begin{aligned}
6x^2+13x+6&=0\\
(2x+3)(3x+2)&=0\\
2x+3=0 \quad &\text{or} \quad 3x+2=0\\
x=-\tfrac{3}{2} \quad &\text{or} \quad x=-\tfrac{2}{3}.
\end{aligned}
$$

This method is useful only when the factorization is easy to do. Many quadratic expressions, like x^2+x-1 and $3x^2+2x-3$, cannot be factorized into the form $(px+q)(rx+s)$, where p, q, r, s are rational numbers. There are two other methods to solve quadratic equations and these two can solve all quadratic equations. These two methods are completing the square and using the quadratic formula.

The method of completing the square is to change the equation from the form $ax^2+bx+c=0$ to $(x+p)^2=q$. This can be done by dividing the whole equation by a and then we have

$$
\begin{aligned}
x^2+\frac{b}{a}x&=-\frac{c}{a}\\
x^2+2\cdot\frac{1}{2}\cdot\frac{b}{a}x+\left(\frac{b}{2a}\right)^2&=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2\\
\left(x+\frac{b}{2a}\right)^2&=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2
\end{aligned}
$$

and thus $p=\frac{b}{2a}$ and $q=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$. If q is positive, then $x+p=\pm q\Rightarrow x=-p\pm q$. If $q=0$, then $x+p=0$ and hence $x=-p$. If q is negative, since the square of all real numbers is non-negative, the equation has no real roots. The result $x+p=\pm q\Rightarrow x=-p\pm q$ still holds, but the roots will be complex.

Example 2.3.

Solve $x^2+6x-16=0$.

Solution.

$$\begin{aligned}
x^2+6x-16&=0\\
x^2+2\cdot 3\cdot x&=16\\
x^2+2\cdot 3\cdot x+3^2&=16+3^2\\
(x+3)^2&=25\\
x+3&=\pm 5\\
x&=2\text{ or }-8.
\end{aligned}$$

Example 2.4.

Solve $2x^2+4x-9=0$.

Solution.

$$\begin{aligned}
2x^2+4x-9&=0\\
x^2+2x&=\frac{9}{2}\\
x^2+2x+1&=\frac{9}{2}+1\\
(x+1)^2&=\frac{11}{2}\\
x+1&=\pm\frac{\sqrt{22}}{2}\\
x&=-1\pm\frac{\sqrt{22}}{2}.
\end{aligned}$$

From the method of completing the square, we can derive the quadratic formula.

$$\begin{aligned}
ax^2+bx+c&=0\\
x^2+\frac{b}{a}x&=-\frac{c}{a}\quad (\because\\
x^2+2\cdot\frac{b}{2a}\cdot x+\left(\frac{b}{2a}\right)^2&=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2\\
\left(x+\frac{b}{2a}\right)^2&=\frac{b^2}{4a}-\frac{c}{a}\\
x+\frac{b}{2a}&=\pm\sqrt{\frac{b^2-4ac}{4a}}\\
x&=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}\\
x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

Thus we have theorem 2.2.

Theorem 2.2.

If $ax^2+bx+c=0$ and $a\neq 0$, the roots of this equation is given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

This formula can be used to solve all quadratic equations in the form $ax^2+bx+c=0$ with $a\neq 0$. Furthermore, by observing the expression in the square root, i.e. b^2-4ac, we can know if the equation has real roots or not. More about this will be discussed in the next section.

Example 2.5.

Solve $x^2-x-12=0$.

Solution.

We put a = 1, b = –1 and c = –12 to the quadratic formula. Then we have

$$\begin{aligned} x &= \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-12)}}{2(1)} \\ &= \frac{1\pm\sqrt{49}}{2} \\ &= 4 \text{ or } -3. \end{aligned}$$

Example 2.6.

Solve $x^2-2x+2=0$.

Solution.

We put a = 1, b = –2 and c = 2 to the quadratic formula. Then we have

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} \\ &= \frac{2 \pm \sqrt{-4}}{2} \\ &= \frac{2 \pm 2\sqrt{-1}}{2} \\ &= 1 \pm \sqrt{-1}. \end{aligned}$$

- $\sqrt{-1}$ is not a real number and it is defined that $\sqrt{-1} = i$. Thus the answer above can be written as $1 \pm i$. i is an imaginary number and $1 \pm i$ are unreal solutions to the equation.

Examples 2.7 and 2.8 are further examples.

Example 2.7.

Solve $3x^4 - 10x^2 - 8 = 0$.

Solution.

Let $y = x^2$, then the original equation becomes

$$\begin{aligned} 3y^2 - 10y - 8 &= 0 \\ (y-4)(3y+2) &= 0 \\ y = 4 \text{ or } y &= -\frac{2}{3}. \end{aligned}$$

Since $y = x^2 \geq 0$ for real x, the solution $y = -\frac{2}{3}$ is rejected. Then we have $x^2 = y = 4$, which gives $x = \pm 2$.

Example 2.8.

Solve $3^{2x+1} - 28 \cdot 3^x + 9 = 0$.

Solution.

Note that the equation is equivalent to $3 \cdot 3^{2x} - 28 \cdot 3^x + 9 = 0$.

Let $y = 3^x$, then the equation becomes

$$
\begin{aligned}
3y^2 - 28y + 9 &= 0 \\
(y-9)(3y-1) &= 0 \\
y &= 9 \text{ or } \frac{1}{3} \\
3^x &= 9 \text{ or } \frac{1}{3} \\
x &= 2 \text{ or } -1
\end{aligned}
$$

Exercises

1. Solve $x^2 - 10x + 16 = 0$.

2. Solve $x^2 = 7 - 2x$.

3. Solve $(\log x)^2 - \log x^4 + 4 = 0$.

4. Solve $4^x - 6 \cdot 2^y - 16 = 0$.

3. Sum and product of roots

Theorem 3.1.

Let α and β be the roots of a quadratic equation $ax^2 + bx + c = 0$ with $a \neq 0$, then

(1) $\alpha + \beta = -\frac{b}{a}$ and

(2) $\alpha\beta = \frac{c}{a}$.

Proof. From the quadratic formula, we have

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-2b}{2a}$$
$$=-\frac{b}{a}$$

and

$$\alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$
$$=\frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2}$$
$$=\frac{b^2-b^2+4ac}{4a^2}$$
$$=\frac{c}{a}.$$

Q.E.D.

- Alternatively, we can find these two relations by comparing coefficients. Since α and β are roots to the equation, we have

  $$(x-\alpha)(x-\beta)=0$$
  $$x^2-(\alpha+\beta)x+\alpha\beta=0.$$

  The original equation is

  $$ax^2+bx+c=0$$
  $$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

  and by comparing coefficients, we have $-(\alpha+\beta)=\frac{b}{a}\Rightarrow\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

- Note that the relation of sum and product of roots with the coefficients always holds, no matter if the roots are real or not.

Example 3.1.

Let α and β be the roots of $2x^2+4x-3=0$ and $\alpha>\beta$. Find the values of $\alpha\beta$, $(\alpha+\beta)^2$ and $\alpha-\beta$.

Solution.

From theorem 3.1, we have

$$\alpha+\beta=-\frac{4}{2}=-2$$
$$\alpha\beta=\frac{-3}{2}=-\frac{3}{2}.$$

Thus $(\alpha+\beta)^2=(-2)^2=4$.

Note that

$$\begin{aligned}(\alpha-\beta)^2&=\alpha^2+\beta^2-2\alpha\beta\\&=\alpha^2+2\alpha\beta+\beta^2-4\alpha\beta\\&=(\alpha+\beta)^2-4\alpha\beta\\&=4-4\left(-\frac{3}{2}\right)\\&=10\end{aligned}$$

Since $\alpha>\beta$, we have $\alpha-\beta=\sqrt{(\alpha-\beta)^2}=\sqrt{10}$.

Theorem 3.2.

Given the sum of roots and product of roots, we can form a quadratic equation whose roots have the required sum and product and this equation is

$$x^2-(\text{sum of roots})x+(\text{product of roots})=0.$$

Example 3.2.

Let α and β be roots of $2x^2+3x+4=0$. Form a quadratic equation, with integer coefficients, whose roots are $2\alpha\beta$ and $\alpha^2+\beta^2$.

Solution.

We form the quadratic equation by finding the sum and product of roots of the required equation.

For $2x^2+3x+4=0$,

$$\alpha+\beta=-\frac{3}{2}$$
$$\alpha\beta=\frac{4}{2}=2.$$

For the required equation,

$$\begin{aligned}\text{sum of roots} &= 2\alpha\beta + \alpha^2 + \beta^2 \\ &= (\alpha+\beta)^2 \\ &= \left(-\frac{3}{2}\right)^2 \\ &= \frac{9}{4}\end{aligned}$$

and

$$\begin{aligned}\text{product of roots} &= (2\alpha\beta)\left(\alpha^2 + \beta^2\right) \\ &= 2\alpha\beta\left[\left(\alpha^2 + 2\alpha\beta + \beta^2\right) - 2\alpha\beta\right] \\ &= 2\alpha\beta\left[(\alpha+\beta)^2 - 2\alpha\beta\right] \\ &= 2\cdot 2\left[\left(-\frac{3}{2}\right)^2 - 2\cdot 2\right] \\ &= -7.\end{aligned}$$

Thus the required equation is

$$\begin{gathered}x^2 - \frac{9}{4}x - 7 = 0 \\ 4x^2 - 9x - 28 = 0.\end{gathered}$$

Exercises.

1. Given α and β are the roots of $2x^2 + 6x + 3 = 0$. Find
   (a) $\alpha + \beta$
   (b) $\alpha\beta$
   (c) $\alpha^2 + \beta^2$
   (d) $\alpha^3 - \beta^3$

2. Given α and β are the roots of $x^2 + 5x + 3 = 0$. Find an equation with integer coefficients whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

4. Discriminant and nature of roots

As we can see from the quadratic formula, the term inside the square root sign, i.e. $b^2 - 4ac$, determines the nature of roots.

Definition 4.1.

For a quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, the **discriminant**, usually denoted as Δ, is defined as

$$\Delta = b^2 - 4ac .$$

The value of the discriminant tells us the number of real roots of the quadratic equation.

Theorem 4.1.

For a quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, and its discriminant is $\Delta = b^2 - 4ac$,

(1) if $\Delta > 0$, the equation has two distinct real roots.

(2) if $\Delta = 0$, the equation has one double real root.

(3) if $\Delta < 0$, the equation has no real root, it has two distinct unreal roots.

Example 4.1.

Find the number of real root(s) for

(1) $x^2 + 6x + 7 = 0$

(2) $x^2 + 6x + 16 = 0$

(3) $x^2 + 6x + 9 = 0$

Solution.

We first calculate the value of the discriminant of the equation and then we can determine the number of real root(s).

For (1), $\Delta = 6^2 - 4(1)(7) = 8 > 0$, thus there are two distinct real roots.

For (2), $\Delta = 6^2 - 4(1)(16) = -28 < 0$, thus there is no real root.

For (3), $\Delta = 6^2 - 4(1)(9) = 0$, thus there is one real root.

Example 4.2.

Find the range of values of k such that $x^2 - kx + 9 = 0$ has real root(s).

Solution.

For the equation to have real root(s), $\Delta \geq 0$.

Thus we have

$$(-k)^2 - 4(1)(9) \geq 0$$
$$k^2 \geq 36$$
$$k \leq -6 \text{ or } k \geq 6.$$

Exercises.

1. Find the range of values of k such that $x^2 - 2kx + k = -6$ has two distinct real roots.

5. Quadratic functions and their graphs

A quadratic function is a function in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. The graph of a quadratic function is a parabola. If $a > 0$, the parabola opens upward; if $a < 0$, the parabola opens downward. These two cases are shown in Figure 1 and Figure 2 respectively.

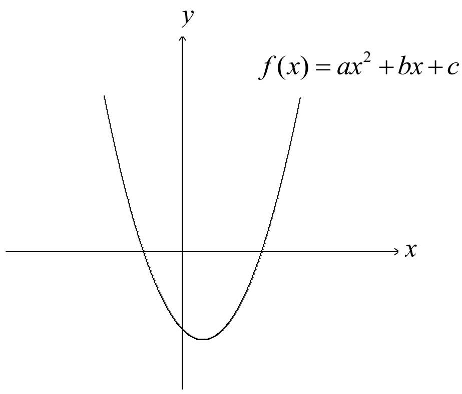

Figure 1: The graph of $f(x) = ax^2 + bx + c$, where $a > 0$

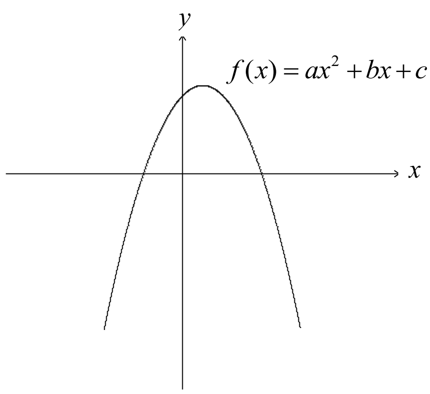

Figure 2: The graph of $f(x) = ax^2 + bx + c$, where $a < 0$

For a parabola, it has a vertex and an axis of symmetry. The coordinates of the former and the equation of the latter can be found from $f(x) = ax^2 + bx + c$ by completing the square.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left[x^2 + 2\cdot\frac{b}{2a}\cdot x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{2a}\right] \end{aligned}$$

If a is positive, since $(x + \frac{b}{2a})^2 \geq 0$ for all real x, a and b, $f(x)$ attains its minimum when $x + \frac{b}{2a} = 0 \Rightarrow x = -\frac{b}{2a}$. If a is negative, since $(x + \frac{b}{2a})^2 \geq 0$, this gives $a(x + \frac{b}{2a})^2 \leq 0$ and hence $f(x)$ attains its maximum when $x + \frac{b}{2a} = 0 \Rightarrow x = -\frac{b}{2a}$.

For both cases, the vertex is $(-\frac{b}{2a}, \frac{4ac-b^2}{2})$ and the axis of symmetry is $x = -\frac{b}{2a}$. The vertex corresponds to the maximum or minimum value of the function.

Example 5.1.

Find the maximum value or minimum value, if exist, of $f(x) = x^2 + 4x + 9$ and $g(x) = -2x^2 + 3x - 1$.

Solution.

$$\begin{aligned} f(x) &= x^2 + 4x + 9 \\ &= (x^2 + 4x + 4) + 5 \\ &= (x + 2)^2 + 5 \end{aligned}$$

Thus the minimum value of $f(x)$ is 5.

$$\begin{aligned} g(x) &= -2x^2 + 3x - 1 \\ &= -2\left(x^2 - \frac{3}{2}x\right) - 1 \\ &= -2\left[x^2 - \frac{3}{2}x + \left(\frac{1}{2}\cdot\frac{3}{2}\right)^2\right] - 1 + 2\left(\frac{1}{2}\cdot\frac{3}{2}\right)^2 \\ &= -2\left(x + \frac{3}{4}\right)^2 + \frac{1}{8} \end{aligned}$$

Thus the maximum value of $g(x)$ is $\frac{1}{8}$.

Exercises.

1. For each of the following functions, find the maximum or minimum value, if exists. Find also the value of x when the maximum or minimum is attained.
   (a) $f(x) = x^2 + 4x - 4$
   (b) $g(x) = -3x^2 + 6x + 4$

The graph of a quadratic function may cut the x-axis at two points, one point or it may not intersect with the x-axis, depending on the sign of the discriminant. For $f(x) = ax^2 + bx + c$, the value of the discriminant $\Delta = b^2 - 4ac$ tells us the number of real roots of the equation $f(x) = ax^2 + bx + c = 0$, i.e. the number of intersection point(s) of the graph of $f(x)$ and $x = 0$.

If $\Delta > 0$, the equation $f(x) = 0$ has two real roots and thus the parabola of $f(x) = ax^2 + bx + c$ cuts the x-axis at two distinct points. This is shown in Figure 3.

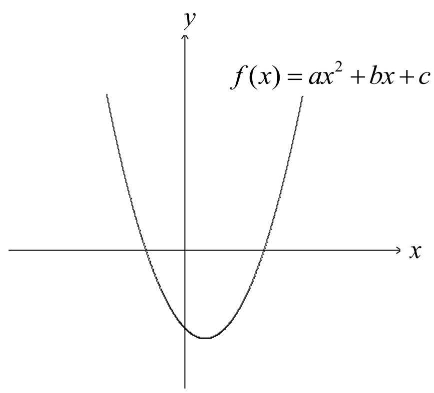

Figure 3: The graph of $f(x) = ax^2 + bx + c$, with $\Delta > 0$

If $\Delta = 0$, the parabola touches the x-axis, for $f(x) = 0$ has only one real root, as shown in Figure 4.

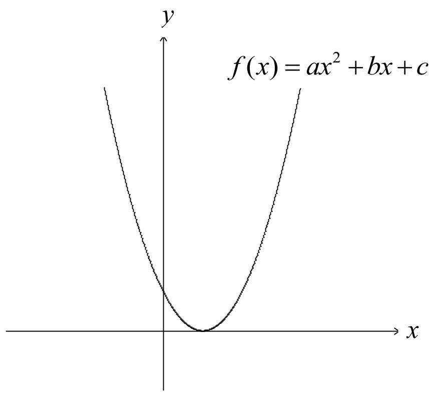

Figure 4: The graph of $f(x)=ax^2+bx+c$, with $\Delta = 0$

If $\Delta < 0$, $f(x)=0$ has no real root and hence the graph of $f(x)$ does not intersect with the x-axis. The whole parabola lies above the x-axis (for $a > 0$, as shown in Figure 5) or below the x-axis (for $a < 0$, as shown in Figure 6).

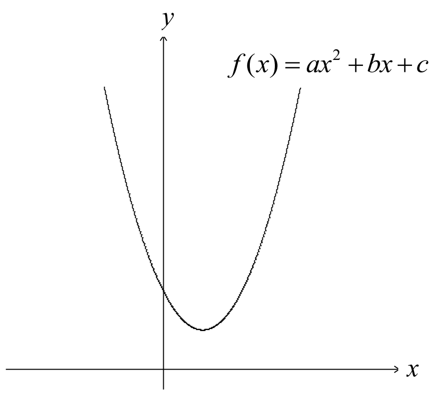

Figure 5: The graph of $f(x)=ax^2+bx+c$, with $\Delta < 0$ and $a > 0$

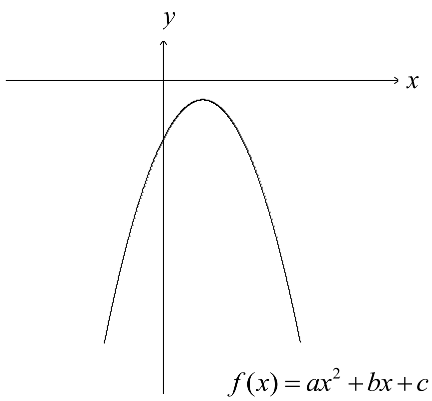

Figure 6: The graph of $f(x)=ax^2+bx+c$, with $\Delta < 0$ and $a < 0$

Exercises

2. Show that the graph of $f(x) = (k+6)x^2 - kx + 2$ must cut the x-axis at at least one point for all real k.

6. Solving a linear equation and a quadratic equation

Given a system of two equations, one being linear and the other being quadratic, in the form

$$\begin{cases} y = px + q \\ y = ax^2 + bx + c \end{cases},$$

we can solve it by substituting the linear equation to the quadratic one to give

$$px + q = ax^2 + bx + c$$
$$ax^2 + (b-p)x + (c-q) = 0.$$

Then we can solve for x and hence y.

The number of real solutions to the system of equations can be determined by finding the discriminant of the quadratic equation $ax^2 + (b-p)x + (c-q) = 0$.

Furthermore, these two equations can be solved by drawing the graphs of the straight line $y = px + q$ and the parabola $y = ax^2 + bx + c$. The point(s) of intersection, if any, give(s) the solution. The discriminant also tells the number of intersecting point(s).

Example 6.1.

Find the equation of straight line that passes through (1, 1) and is tangent to the parabola $y = x^2 - 3x + 7$. Find also the point of tangency.

Solution.

Let the equation of the required straight line be $y = mx + c$. Since (1, 1) lies on this line, we have

$$1 = m(1) + c$$
$$c = 1 - m.$$

Putting the equation of the straight line to that of the parabola, we get

$$x^2 - 3x + 7 = mx + c$$
$$x^2 + (-3-m)x + (7-c) = 0.$$

Since the straight line is tangent to the parabola, there is only one point of intersection and hence

$$\begin{aligned} \Delta &= 0 \\ (-3-m)^2 - 4(1)(7-c) &= 0 \\ 9+6m+m^2-28+4c &= 0 \\ m^2+6m+4(1-m)-19 &= 0 \quad (\text{Put } c = 1-m) \\ m^2+2m-15 &= 0 \\ (m+5)(m-3) &= 0 \\ m &= -5 \text{ or } 3. \end{aligned}$$

For $m = -5$, $c = 1 - (-5) = 6$ and so the equation of the straight line required is $y = -5x + 6$.

For $m = 3$, $c = 1 - 3 = -2$ and thus the equation required is $y = 3x - 2$.

Exercises

1.

7. Solutions to Selected Exercise

Solving quadratic equations

1. $x = 8$ or $x = 2$

2. $x = -1 \pm 2\sqrt{2}$

3. $x = 100$

4. $x = 3$

Sum and product of roots

1. (a) -3

   (b) $\frac{3}{2}$

   (c) 6

(d) $\frac{15\sqrt{3}}{2}$

2. $3x^2 - 19x + 3 = 0$

Discriminant and nature of roots

1. $k > 3$ or $k < -2$

Quadratic functions and their graphs

1. (a) The minimum value is –8 when $x = -2$.

   (b) The maximum value is 7 when $x = 1$.

Solving a linear equation and a quadratic equation

1.