How to Solve:
Functions and Custom Characters

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Theory

Definition: What is a Function?

A function is a relation which takes an input and gives a unique output for that input.

Ex: f(x) = 2x + 1
For each value of x we will get a unique value of f(x)
If x=1, then in f(x) we will replace x with 1 on both left and right hand side of f(x) = 2x+1 to get
f(1) = 2*1 + 1 = 3

Domain and Range of a Function

Domain: The set of values which the independent variable [Ex: (x) in f(x)] can take for which the dependent variable [Ex: f(x)] has a valid value [Ex: value should not be undefined]

Ex: f(x) = $\frac{1}{x-2}$
Now, we know that a fraction becomes undefined when the denominator is zero. So, f(x) = $\frac{1}{x-2}$ will become undefined when denominator will become 0
So, for x-2 = 0 or, x=2 f(x) is not defined

So, Domain of f(x) is $x \in R$ (x belongs to all real numbers) and $x \neq 2$

Range: The set of resulting values of the function f(x) for all possible values of x in the domain of f(x)

Ex: f(x) = $\frac{1}{x-2}$
Range of f(x) is the set of all values which f(x) takes when x takes the values in the domain (i.e $x \in R$ and $x \neq 2$) of f(x)
Ex. x=3 is in domain so, f(3) = $\frac{1}{3-2}$ = 1 is in Range of the function.

Problem Types

PT1 : f(constant) Problems

In this type of problems f(x) will be given and we will be asked to find the value of f(constant), ex f(1) and so no. Let's take some examples:

Q1. f(x) = 3x + 1. Find f(2)

Solution:
Compare the things inside the bracket in f(x) and f(2). So, we need to replace x with 2 in f(x) = 3x + 1 to get the value of f(2) => f(2) = 3*2 + 1 = 7

Q2. If f(x) = $\frac{x+3}{2x-6}$, where $x \neq 3$, then find f(5).

Solution:
Replace x with 5 in f(x) = $\frac{x+3}{2x-6}$we get
f(5) = $\frac{5+3}{2*5-6}$= $\frac{8}{4}$ = 2

Q3. If f(x) =2x^2 + 5 and f(a) = 13. Then find the value of a.

Solution:
Replace x with a in f(x) =2x^2 + 5 we have
f(a) =2a^2 + 5 = 13 (given)
=> 2a^2 = 13 - 5 = 8
=> a^2 = 4 => a = ± 2

Q4. If f(x) = $\frac{5x-1}{3x-7}$, where $x \neq \frac{7}{3}$, and f(a) = 7 then find the value of a.

Solution:
Replace x with a in f(x) = $\frac{5x-1}{3x-7}$ we have
f(a) = $\frac{5a-1}{3a-7}$ = 7 (given)
=> 5a - 1 = 7* (3a-7) => 5a - 1 = 21a – 49

=> 48 = 16a => a = 3

PT2 : Reciprocal Problems

In this type of problems we will be given f(x) and will be required to find the value of f(1x). Let's take some examples:

Q1. If f(x) = 2x+1, then find the value of f($\frac{1}{x}$)

Solution:
We will compare what is inside the bracket of f(x) and f($\frac{1}{x}$).
So, we need to replace x with $\frac{1}{x}$ in f(x) = 2x+1 to get the value of f($\frac{1}{x}$)
=> f($\frac{1}{x}$) = 2*$\frac{1}{x}$+1 = $\frac{2+x}{x}$

Q2. If f(x) =$\frac{x-2}{3x+4}$ then find the value of f($\frac{1}{x}$)

Solution:
Replace x with $\frac{1}{x}$ in f(x) = $\frac{x-2}{3x+4}$,we get
f($\frac{1}{x}$) = $\frac{\frac{1}{x}-2}{3\frac{1}{x}+4}$
=> f($\frac{1}{x}$) = $\frac{1-2x}{3+4x}$

PT3 : Nested Functions Problems

In this type of problems we will be given two functions, let's say f(x) and g(x) and will be required to find the value of f(g(x)) or g(f(x)). Let's take some examples:

Q1. If f(a) =a^2, g(a) = a^3, Find f(g(a))

Solution:
f(g(a)) is called as nested function, as one function is inside the other.
Let's start by finding the value of g(a) first. g(a) = a^3 [given]
So, f(g(a)) = f(a^3)
To find f(a^3) we will replace a with a^3 in f(a) =a^2 to get
f(a^3) = ((a^3)^2) = a^6
=> f(g(a)) = a^6

Q2. If f(x) = 2x-3 and g(x) = x^3. Then find f(g(x))

Solution:
Let's start by finding the value of g(x) first. g(x) = x^3 [given]
So, f(g(x)) = f(x3) = 2*x^3 - 3

Q3. If f(x) = $\frac{2x+3}{3x+5}$ and g(x) = 3x – 2, then find the value of f(g(2)) and g(f(3))

Solution:
f(g(2)) = f(3*2 -2) = f(4) = $\frac{2*4+3}{3*4+5}$ = $\frac{11}{17}$

g(f(3)) = g($\frac{2*3+3}{3*3+5}$) = g($\frac{9}{14}$) = 3*$\frac{9}{14}$ - 2 = $\frac{27-28}{14}$ = $\frac{-1}{14}$

Q4. If f(x) = 3x-2 and g(x) = x^2. Then for what value of x is f(g(x)) = g(f(x))

Solution:
f(g(x)) = f(x^2) = 3*x^2 - 2
g(f(x)) = g(3x-2) = $(3x-2)^2$

f(g(x)) = g(f(x)) => 3*x^2 - 2 = $(3x-2)^2$
=> 3*x^2 - 2 = 9x^2 -2*3x*2 +2^2 [Using $(a-b)^2 = a^2 - 2ab + b^2$]
=> 3*x^2 - 2 = 9x^2 -12x +22
=> 6x^2 - 12x + 6 = 0 [Dividing both the sides by 6]
=> x^2 - 2x + 1 = 0
=> $(x-1)^2$ = 0
=>x = 1

PT4 : Simultaneous Equations Problems

In this type of problems we will be given a function and some of the values for that function and we would be asked to find value of f(something). Let's take some examples:

Q1. If f(x) = a + bx and f(1) = 6 and f(2) = 10, then find the value of f(10)

Solution:
f(1) = a + b*1 = a + b = 6 (given) ....(1)
f(2) = a + b*2 = a + 2b = 10 (given) .....(2)

(2) - (1) we get
(a+2b) - (a+b) = 10 - 6 =4
=> b= 4
a + b = 6 => a = 2
So, f(x) = 2 + 4x
f(10) = 2 + 4*10 = 42

Q2. If $f(x) = ax^2 + bx$ and f(1)=10 and f(2)=30, then find the value of f(3)

Solution:
$f(1) = a1^2 + b*1 = a + b = 10$ (given) ....(1)
$f(2) = a2^2 + b*2 = 4a + 2b = 30$ (given) ....(2)

(1) * 4 - (2) we get
4(a+b) - (4a + 2b) = 4*10 - 30 = 10
2b = 10 => b = 5
a + b = 10 => a = 5
$f(x) = 5x^2 + 5x$
$f(3) = 5*3^2 + 5*3 = 45 + 15 = 60$

PT5 : Symmetric Functions Problems

In this type of problems we will be given couple of f(x)'s and we will be asked for which of this function f(X) = f(1-x). Lets take some examples:

Q1. For which of the following functions is f(x) = f(1-x) for all values of x ?

A. $f(x) = x^2$
B. f(x) = x + 1
C. f(x) = 2x + 1
D. f(x) = x*(1-x)

Solution:
We need to find the value of f(1-x) using f(x) in all the four options and check for which option is the value of f(1-x) exactly equal to f(x)
A. $f(x) = x^2$
$f(1-x) = (1-x)^2 = 1 - 2x + x^2 \neq x^2$ for all values of x. So, f(x) ≠ f(1-x) for all x

B. f(x) = x + 1
f(1-x) = (1-x) + 1 = 2 - x ≠ x + 1 for all values of x. So, f(x) ≠ f(1-x) for all x

C. f(x) = 2x + 1
f(1-x) = 2(1-x) + 1 = 3 - 2x ≠ 2x + 1 for all values of x. So, f(x) ≠ f(1-x) for all x

D. f(x) = x*(1-x)
f(1-x) = (1-x)*(1-(1-x)) =(1-x)*x = x*(1-x) for all values of x. So, f(x) = f(1-x) for all x
So, Answer is D

Now, this is lengthy to do. But, there is a theory behind symmetry which can make these problems easier to solve:

How to check if a function is symmetric in terms of x and 1-x

$f(x) = f(1-x)$ when $f(x)$ is symmetric in terms of x and (1-x), which means
If there is a term of x in numerator then there should be a term of (1-x) in numerator with the same power and the same sign
If there is a term of x in denominator then there should be a term of (1-x) in denominator with the same power and the same sign

Examples of Symmetric Functions

$f(x) = f(1-x)$ in the below examples as the function is symmetric in terms of x and (1-x)

1. $f(x) = x^2 * (1-x)^2$
Function is having a term of x and a term of (1-x) in the numerator with the same power (2) and the same sign (+)

2. $f(x) = x^2 + (1-x)^2$
Function is having a term of x and a term of (1-x) in the numerator with the same power (2) and the same sign (+)

3. $f(x) = \frac{1}{-x\text{^}3-(1-x)\text{^}3}$
Function is having a term of x and a term of (1-x) in the denominator with the same power (3) and the same sign (-)

Examples of Non-Symmetric Functions

$f(x) \neq f(1-x)$ in the below examples as function is non-symmetric in terms of x and (1-x)

1. $f(x) = x^2 * (1-x)^3$
Different Powers: Function is having a term of x and a term of (1-x) in the numerator and the same sign (+) but the power of x and (1-x) is different, 2 and 3 respectively

2. $f(x) = x^2 - (1-x)^2$
Different Signs: Function is having a term of x and a term of (1-x) in the numerator with the same power (2) but different sign (+) and (-) respectively

3. $f(x) = \frac{1}{-x3+(1-x)\text{^}3}$

Function is having a term of x and a term of (1-x) in the denominator with the same power (3) but different signs

4. $f(x) = \frac{x\text{^}3}{(1-x)\text{^}3}$
Function is having a term of x in numerator but no term of (1-x) in numerator and a term for (1-x) in denominator but no term of x in denominator, so not symmetric

PT6 : Custom Characters Problems

In this type of problems we will be given some custom characters like @ their definition will be given too. We will to consider them as functions and solve the problems. Lets take some examples:

Q1. If a@b = a^2 - 2ab, then find the value of 2 @ (3@1)

Solution:
Here we will consider @ a function of a and b
Let's find the value of 3@1 first.
we will compare 3@1 with a@b => a= 3 and b= 1
=> 3@1= 3^2 - 2*3*1 = 9 - 6 = 3
=> 2 @ (3@1) = 2 @ 3 = 2^2 - 2*2*3 = 4 - 12 = -8

Q2. If a □ b = b □ a for all values of a and b, then what can be the possible operation □ ?

1. Addition
2. Multiplication
3. Division
4. Subtraction

Solution:
1. Addition => a + b = b + a which is true for all values of a and b. So, it is possible
2. Multiplication => a * b = b * a which is true for all values of a and b. So, it is possible
3. Division=> ab= ba which does not have to be true for all values of a and b. So, it is NOT possible
4. Subtraction => a - b = b - a which does not have to be true for all values of a and b. So, it is NOT possible

So, answer is a and 2

Q3. If a # b = a + b - 2ab, then find the value of (4 # 2) # (1 # 3)

1. 2 # 3
2. 3 # 4
3. 4 # 8
4. 1 # 2

Solution:
4 # 2 = 4 + 2 - 2*4*2 = -10
1 # 3 = 1 + 3 - 2*1*3 = -2
=> (4 # 2) # (1 # 3) = -10 # -2 = -10 -2 -2*(-10)*(-2) = -12 - 40 = -52
Now, the answer choices are also given in terms of the custom characters. So, we need to solve each answer to check which one will give us -52 answer.
1. 2 # 3 = 2+3 - 2*2*3 = 5 - 12 = -7, so not the answer
2. 3 # 4 = 3+4 - 2*3*4 = 7 - 24 = -17, so not the answer
3. 4 # 8 = 4+8 - 2*4*8 = 12 - 64 = -52, so THIS is the answer. We don't need to solve further but solving just to explain the process
4. 1 # 2 = 1+2 - 2*1*2 = 3 - 4 = -1

Q4. Solve for ◊ and □ in the below subtraction equation. (given that ◊ and □ are single digit numbers)

◊ ◊ 4
- □ ◊
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5 7 □

Solution:
Check the Video for explanation
Answer is ◊ = 6 and □ = 8

Q5. Solve for ◊ and □ in the below multiplication equation. (given that ◊ and □ are single digit numbers)

3 ◊
*** □ 2**
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15 5 □

Solution:
Check the Video for explanation
Answer is ◊ = 7 and □ = 4