9

Properties of the Circle

TERMINOLOGY

Arc: Part of a curve, most commonly a portion of the distance around the circumference of a circle

Chord: A straight line joining two points on the circumference of a circle

Concentric circles: Circles that have the same centre

Concyclic points: Points that lie on the circumference of the same circle

Cyclic quadrilateral: A cyclic quadrilateral is a figure whose four vertices are concyclic points. The four vertices lie on the circumference of a circle

Radius: A radius is the distance from the centre of a circle out to the circumference (radii is plural, meaning more than one radius)

Subtend: Form an angle at some point (usually the centre or circumference of a circle)

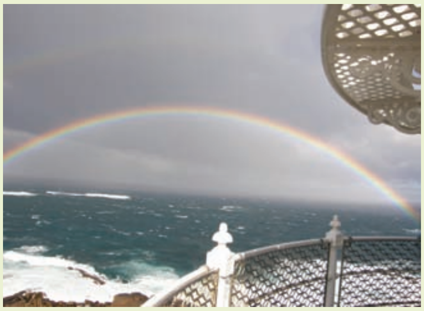
Tangent: A straight line external to a curve or circle that just touches the curve or circle at a single point

INTRODUCTION

IN CHAPTER 4, YOU STUDIED the geometry of angles, triangles, quadrilaterals and other polygons. This chapter shows you some properties of the circle.

DID YOU KNOW?

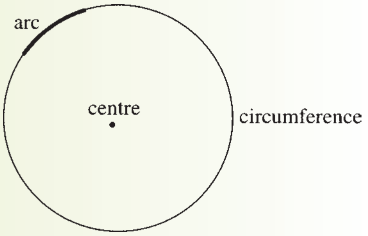
A rainbow is the shape of an arc of a circle. If you could see the whole rainbow, it would form a circle. Research the rainbow on the Internet and find out more about its shape and other properties.



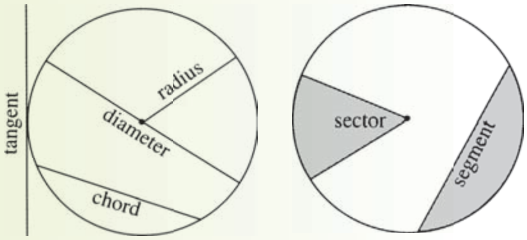
Parts of a Circle

EXTENSION

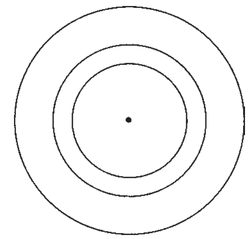
An **arc** is a part of the **circumference**.



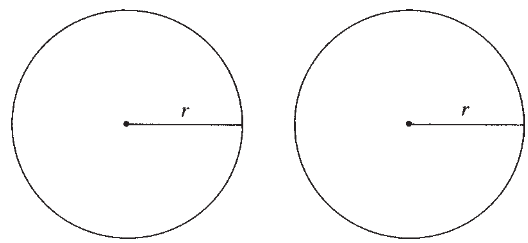
A **tangent** touches the circle at **one point**.



Concentric circles are circles that have the same **centre**.



Equal circles have the same **radius**.



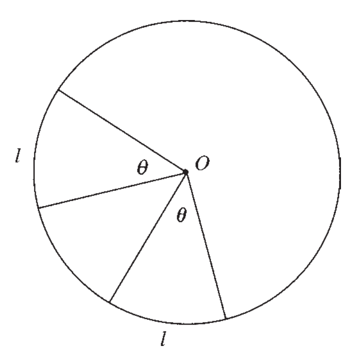
DID YOU KNOW?

Leonardo da Vinci (1452–1519) was a great artist, scientist and inventor. He studied geometry, and many of his model drawings show this influence. His drawings included designs for flying machines, spring-driven automobiles, bridges and weapons. Leonardo's designs were revolutionary, and the scientists of his time did not have the knowledge needed to make the models work.

Arcs, Angles and Chords

EXTENSION

Equal arcs subtend equal angles at the centre of the circle.



Proof

Let two equal arcs have lengths l_1 and l_2, and subtend angles of α and β at the centre of the circle.

Using $l = r\theta$, $l_1 = r\alpha$ and $l_2 = r\beta$

You will study the formula $l = r\theta$ in Chapter 5 of the HSC Course book.

But $l_1 = l_2$

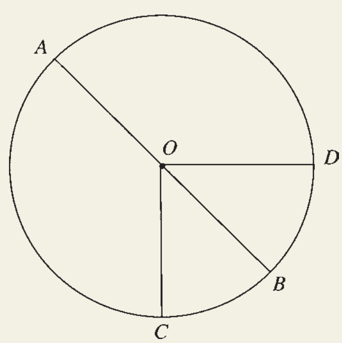
$\therefore \quad r\alpha = r\beta$

$\therefore \quad \alpha = \beta$

The converse is also true:

> If two arcs subtend equal angles at the centre of the circle, then the arcs are equal.

EXAMPLE



AB is a diameter of the circle with centre O. Arc CB = arc BD.
Prove $\angle AOC = \angle AOD$.

Solution

Since arc CB = arc BD, $\angle COB = \angle DOB$

Let $\angle COB = \angle DOB = x$

Then $\angle AOC = 180° - \angle COB$ ($\angle AOB$ is a straight angle)

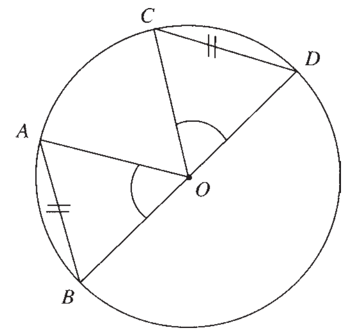
$= 180° - x$

Also $\angle AOD = 180° - \angle DOB$ (similarly)

$= 180° - x$

$\therefore \quad \angle AOC = \angle AOD$

Equal chords subtend equal angles at the centre of the circle.



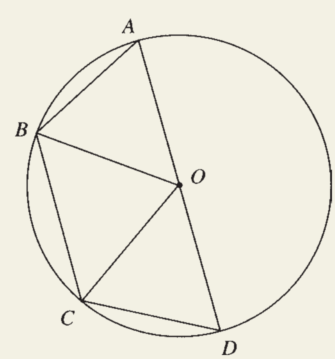
Proof

$$
\begin{aligned}
OA &= OC && \text{(equal radii)}\\
OB &= OD && \text{(similarly)}\\
AB &= CD && \text{(given)}\\
\therefore \text{ by SSS, } \Delta OAB &\equiv \Delta OCD &&\\
\therefore \quad \angle AOB &= \angle COD && \text{(corresponding } \angle\text{s in congruent } \Delta\text{s)}
\end{aligned}
$$

The converse in also true:

Equal angles subtended at the centre of the circle cut off equal chords.

EXAMPLE



AD is a diameter of the circle with centre O, where $AB = CD$. Prove that $BC \parallel AD$.

Solution

Since $AB = CD$, $\angle AOB = \angle COD$

Let $\angle AOB = \angle COD = x$

Then $\angle BOC = 180^\circ - (x + x)$ ($\angle AOD$ is a straight $\angle$)

$= 180^\circ - 2x$

$OB = OC$ (equal radii)

$\therefore$ ΔOBC is isosceles with $\angle OBC = \angle OCB$

$\angle OBC + \angle OCB + 180^\circ - 2x = 180^\circ$ ($\angle$ sum of ΔOBC)

$\angle OBC + \angle OCB = 2x$

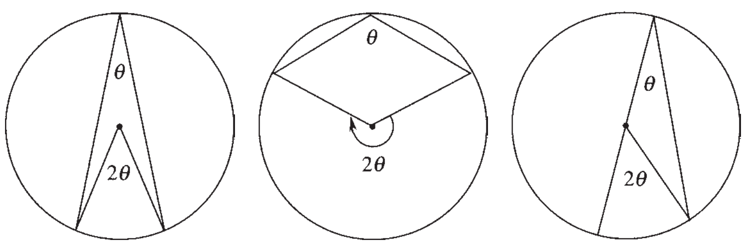
$\therefore$ $\angle OBC = \angle OCB = x$

$\therefore$ $\angle OBC = \angle AOB$

But these are equal alternate angles

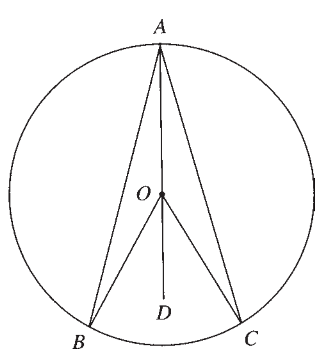
$\therefore$ $BC \parallel AD$

The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



These figures show that this property can look quite different in different situations.

Proof



Join AO and produce to D.

Let $\angle BAO = x$ and $\angle CAO = y$

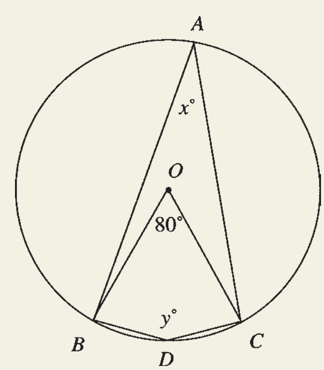
$\therefore$ $\angle BAC = x + y$

$OA = OB$ (equal radii)

$$
\begin{aligned}
\therefore\ \angle OBA &= x && \text{(base }\angle\text{s of isosceles }\Delta) \\
&= \angle BAO \\
OA &= OC && \text{(equal radii)} \\
\therefore\ \angle OCA &= y && \text{(base }\angle\text{s of isosceles }\Delta) \\
&= \angle CAO \\
\angle BOD &= x + x && \text{(exterior }\angle\text{ of }\Delta OBA) \\
&= 2x \\
\angle COD &= y + y && \text{(exterior }\angle\text{ of }\Delta OCA) \\
&= 2y \\
\angle BOC &= \angle BOD + \angle COD \\
&= 2x + 2y \\
&= 2(x + y) \\
&= 2\angle BAC
\end{aligned}
$$

EXAMPLES

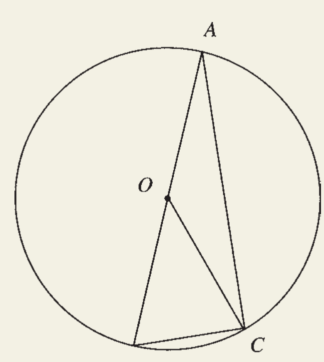
1. Find the values of x and y.



Solution

$$
\begin{aligned}
x &= 40 && (\angle\text{ at centre is twice the }\angle\text{ at the circumference}) \\
\text{Reflex } \angle BOC &= 360^\circ - 80^\circ && (\angle\text{ of revolution}) \\
&= 280^\circ \\
\therefore \quad y &= 140 && (\angle\text{ at centre is twice the }\angle\text{ at the circumference})
\end{aligned}
$$

2. Prove $\angle BOC$ is twice the size of $\angle OCA$.



Solution

Let $\angle OAC = x$

$\angle BOC = 2\angle OAC$ ($\angle$ at centre is twice the $\angle$ at the circumference)

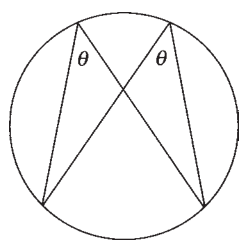
$= 2x$

$OA = OC$ (equal radii)

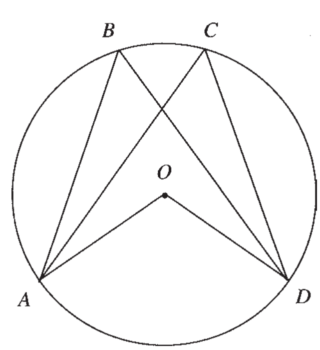
$\therefore$ $\angle OCA = \angle OAC = x$ (base $\angle$s of isosceles Δ)

$\therefore$ $\angle BOC = 2\angle OCA$

Angles in the same segment of a circle are equal.



Proof



Join A and D to centre O

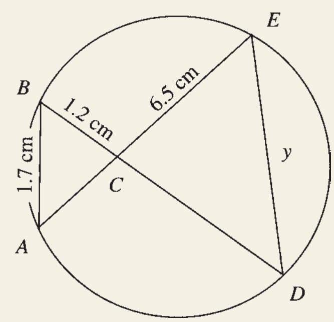
$\angle AOD = 2\angle ABD$ ($\angle$ at centre is twice the $\angle$ at the circumference)

$\angle AOD = 2\angle ACD$

$\therefore$ $\angle ABD = \angle ACD$

EXAMPLE

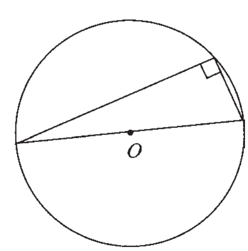
Prove ΔABC and ΔDEC are similar. Hence find the value of y correct to 1 decimal place.



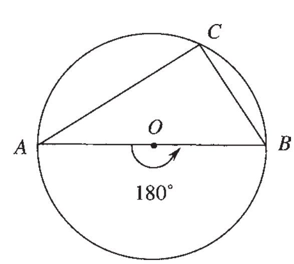
Solution

$$\angle ABC = \angle DEC \quad (\angle\text{s in same segment})$$
$$\angle BCA = \angle ECD \quad (\text{vertically opposite } \angle\text{s})$$
$$\therefore \Delta ABC \parallel\!| \Delta DEC$$
$$\therefore \frac{ED}{BA} = \frac{EC}{BC}$$
$$\frac{y}{1.7} = \frac{6.5}{1.2}$$
$$1.2y = 1.7 \times 6.5$$
$$y = \frac{1.7 \times 6.5}{1.2}$$
$$= 9.2 \text{ cm correct to 1 decimal place.}$$

The angle in a semicircle is a right angle.



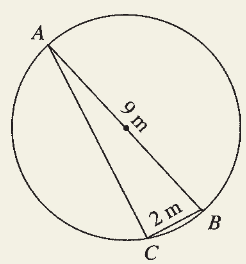
Proof



$\angle AOB = 180^\circ$ (straight $\angle$)
$\angle AOB = 2\angle ACB$ ($\angle$ at centre is twice the $\angle$ at the circumference)
$\therefore \angle ACB = 90^\circ$

EXAMPLE

AB is a diameter of the circle below. If $BC = 2$ m and $AB = 9$ m, find the exact length of AC.



Solution

$\angle ACB = 90^\circ$
$\therefore AB^2 = AC^2 + BC^2$
$9^2 = AC^2 + 2^2$
$81 = AC^2 + 4$
$77 = AC^2$
$\therefore AC = \sqrt{77}$ m

9.1 Exercises

1. Find values of all pronumerals (O is the centre of each circle).

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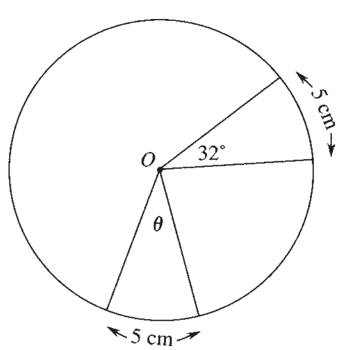

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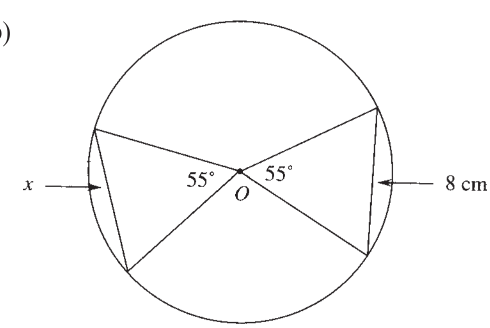
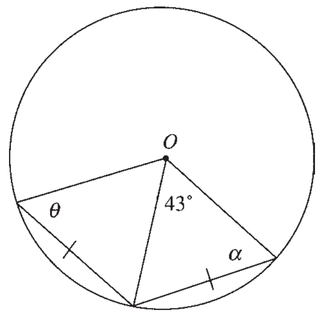


(c)



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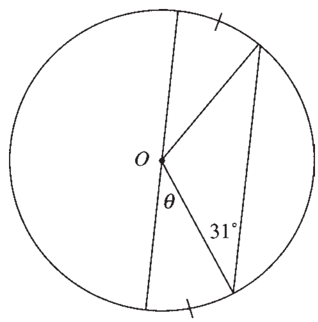

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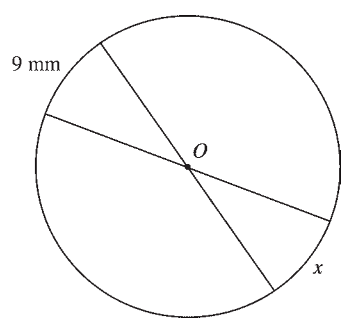

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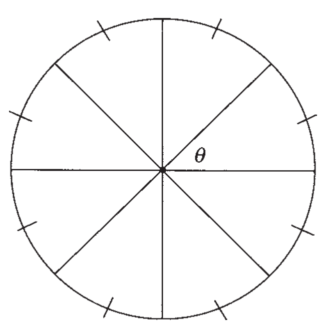
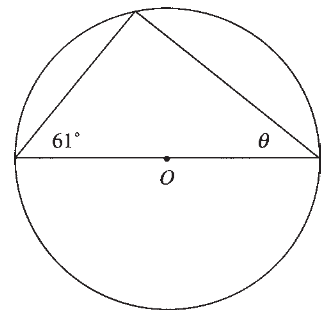


2. The circumference of a circle is 16π cm. Find the length of the arc that subtends an angle of 40° at the centre of the circle.

3. Find values of all pronumerals (O is the centre of each circle).

(a)



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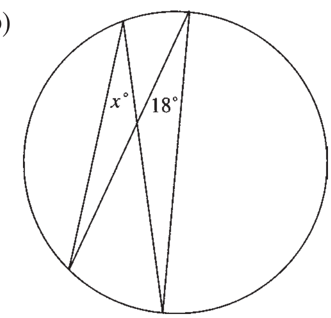

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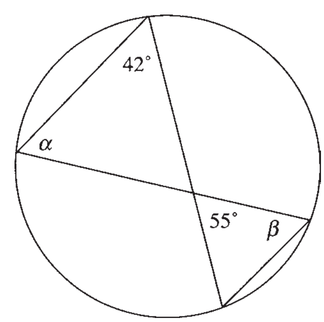

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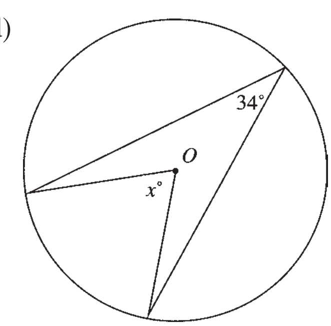

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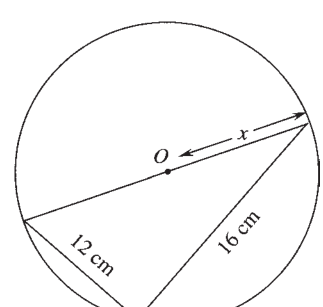

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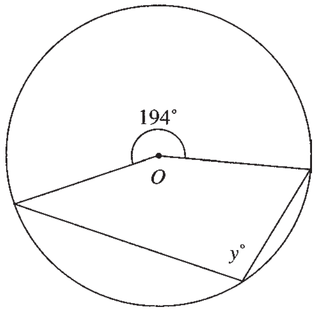

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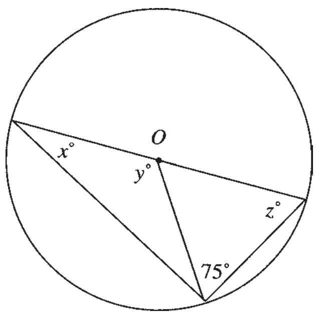

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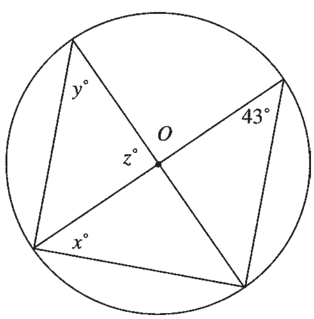
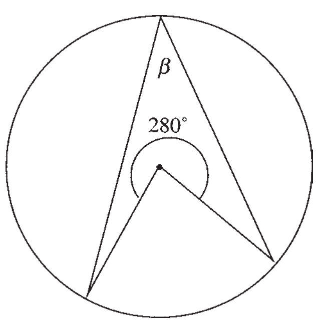


(i)



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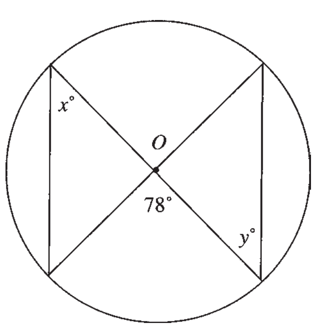
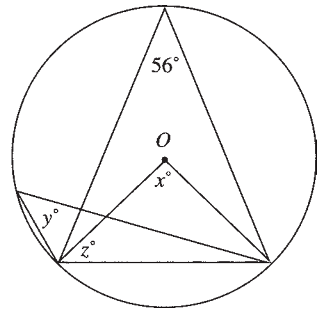


4. Find values of all pronumerals (O is the centre of each circle).

(a)



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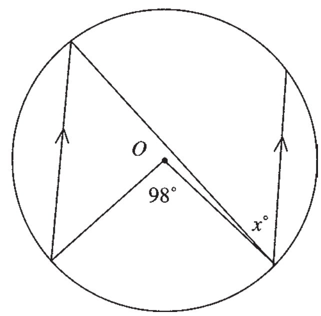

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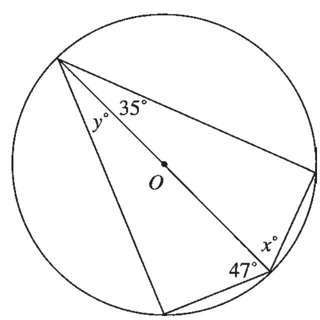

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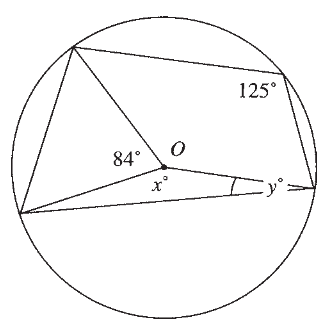

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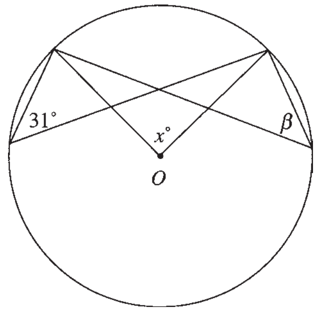

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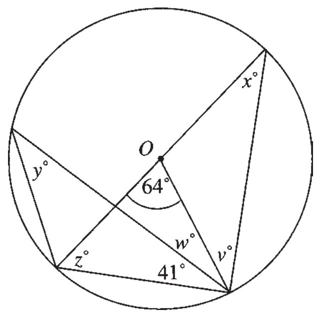

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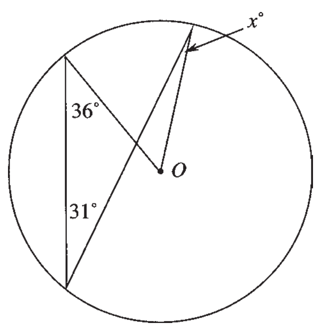

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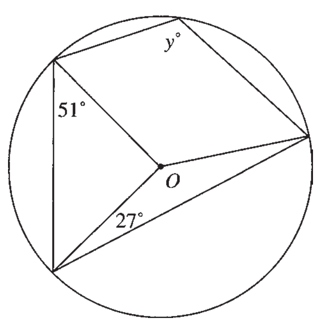

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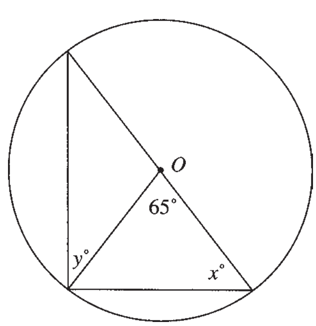

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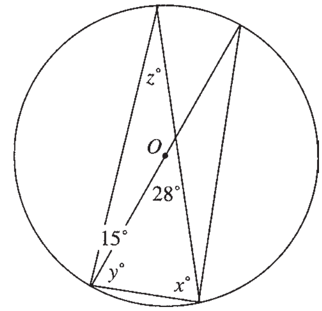
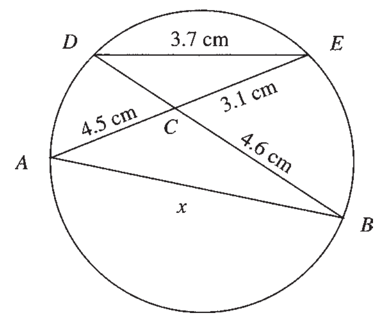
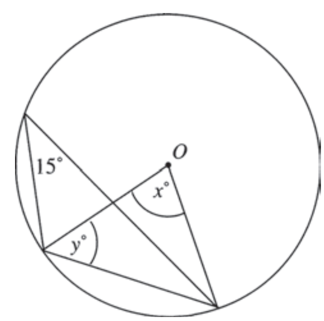


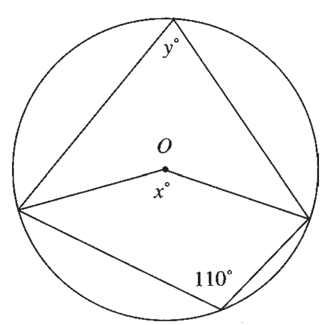
5. (a) Prove $\Delta ABC \mathbin{|||} \Delta DEC$.
(b) Hence find the value of x correct to 1 decimal place.



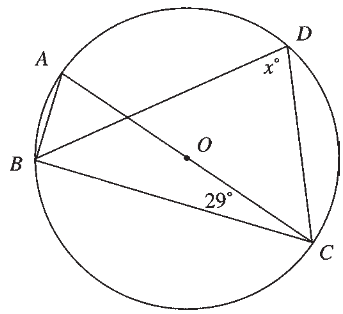
6. Find x and y, giving reasons.



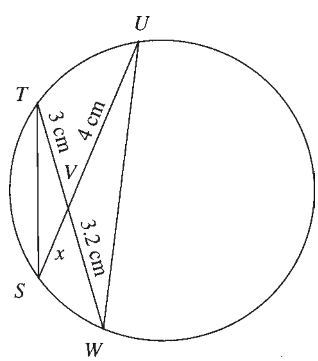
7. Find x and y, giving reasons.



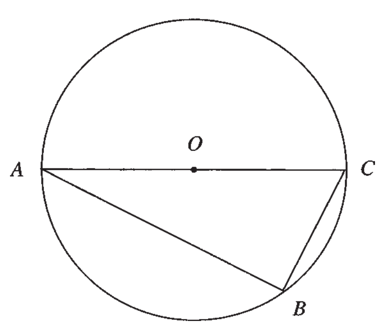
8. Evaluate x, giving reasons for each step in your calculation.



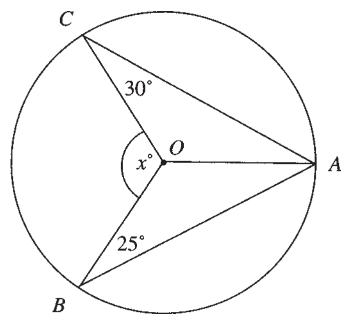
9. Prove ΔSTV and ΔWUV are similar. Hence find x.



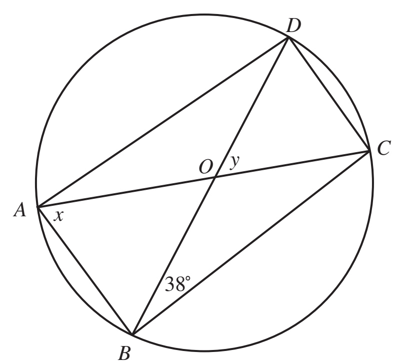
10. $AB = 6$ cm and $BC = 3$ cm. O is the centre of the circle. Show that the radius of the circle is $\dfrac{3\sqrt{5}}{2}$ cm.



11. Find x, giving reasons for each step in your calculations.

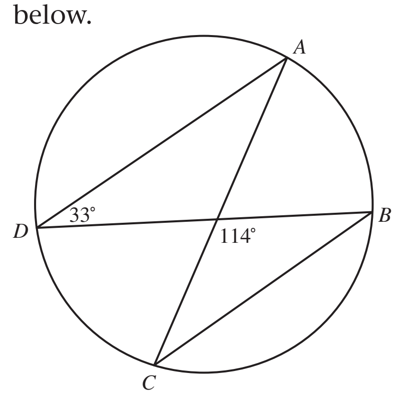


12. The circle below has centre O.

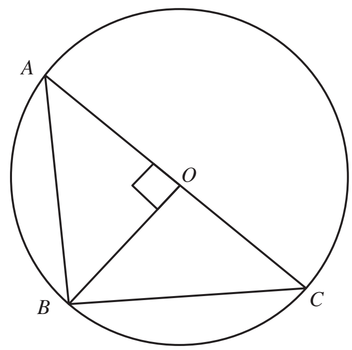


(a) Evaluate x and y.
(b) Show that $AD = BC$.

13. Show that $AD \parallel BC$ in the circle below.

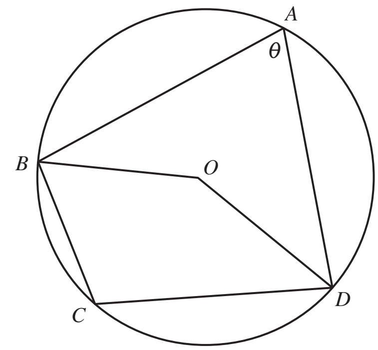


14. A circle has centre O and radius r as shown.



(a) Show that triangles AOB and ABC are similar.
(b) Show that $BC = \sqrt{2}\,r$.

15. The circle below has centre O and $\angle DAB = \theta$.

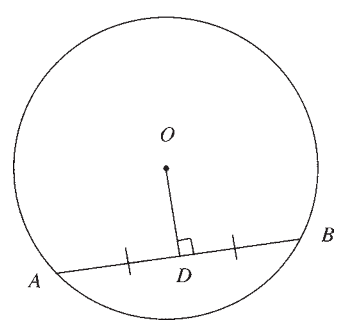


Show that $\angle DAB$ and $\angle BCD$ are supplementary.

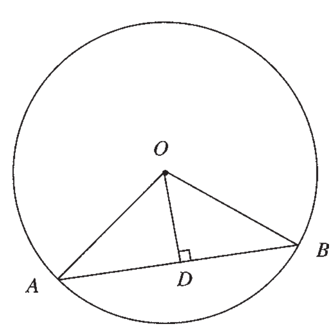
Chord Properties

EXTENSION

> A perpendicular line from the centre of a circle to a chord bisects the chord.



Proof



$\angle ADO = \angle BDO = 90°$ (given)

$OA = OB$ (equal radii)

OD is common

$\therefore$ by RHS $\Delta OAD \equiv \Delta OBD$

$\therefore$ $AD = BD$ (corresponding sides in congruent Δs)

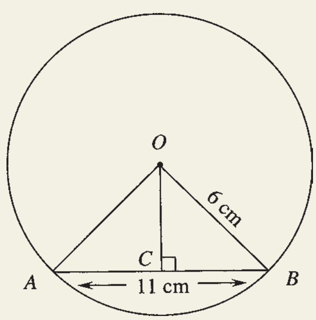
So OD bisects AB

The converse is also true:

> A line from the centre of a circle that bisects a chord is perpendicular to the chord.

EXAMPLES

1. Line OC is perpendicular to chord AB. If the radius of the circle is 6 cm and the chord is 11 cm long, find the length of OC, correct to 1 decimal place.



Solution

$AB = 11$

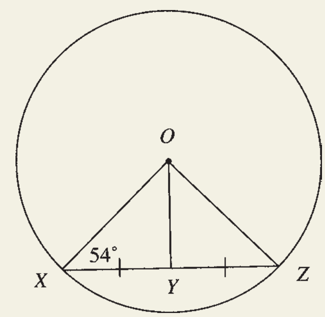
$\therefore$ $AC = 5.5$ (OC bisects AB)

Also $OA = 6$ (radius—given)

$$OA^2 = AC^2 + OC^2$$
$$6^2 = 5.5^2 + OC^2$$
$$36 = 30.25 + OC^2$$
$$5.75 = OC^2$$
$$\therefore\ OC = \sqrt{5.75}$$
$$= 2.4 \text{ cm}$$

CONTINUED

2. Given $XY = YZ$ and $\angle OXY = 54°$, find $\angle XOY$.



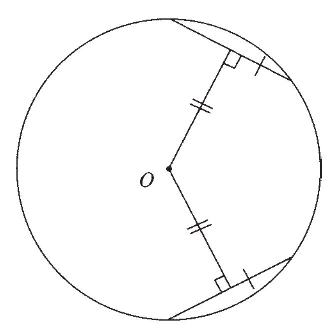
Solution

$\angle OYX = 90°$ (OY bisects XZ)

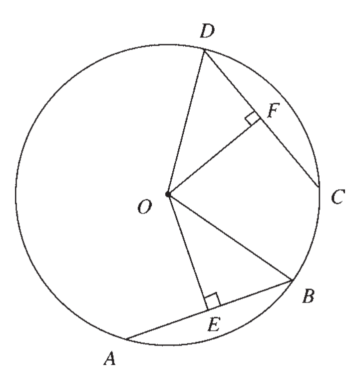
$\angle XOY = 180° - (90° + 54°)$ ($\angle$ sum of ΔOXY)

$= 36°$

Equal chords are equidistant from the centre of the circle.



Proof



Let $CD = AB$

$\angle OEB = \angle OFD = 90°$ (given)

$OB = OD$ (equal radii)

$AB = CD$ (given)

$$BE = \frac{1}{2}AB \qquad (OE \text{ bisects } AB)$$

$$DF = \frac{1}{2}CD \qquad (OF \text{ bisects } CD)$$

$\therefore \quad BE = DF$

$\therefore \quad$ by RHS $\Delta OEB \equiv \Delta OFD$

$\therefore \quad OE = OF \qquad$ (corresponding sides in congruent Δs)

The converse is also true:

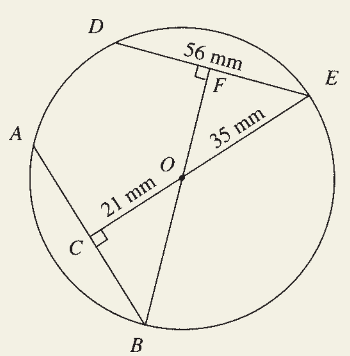
> Chords that are equidistant from the centre are equal.

Class Exercise

Prove that chords that are equidistant from the centre are equal.

EXAMPLE

In the circle below, with centre O, $OE = 35$ mm, $DE = 56$ mm and $OC = 21$ mm. Show that $AB = DE$.



Solution

$EF = 28$ mm $\qquad$ (OF bisects DE)

$OE^2 = EF^2 + OF^2$

$35^2 = 28^2 + OF^2$

$35^2 - 28^2 = OF^2$

$441 = OF^2$

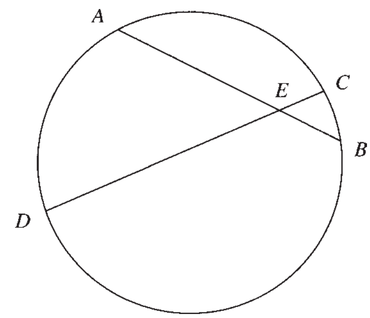
$\therefore \quad OF = \sqrt{441}$

$\qquad = 21$ mm

$\therefore \quad OF = OC \qquad$ (chords equal when equidistant from the centre)

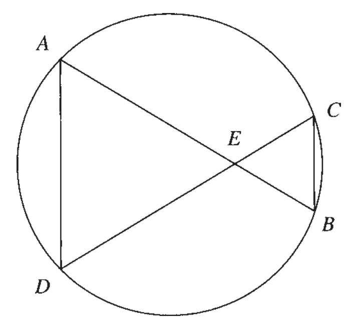
So $\quad AB = DE$.

The products of intercepts of intersecting chords are equal.



$$AE \cdot EB = DE \cdot EC$$

Proof



$\angle AED = \angle CEB$ (vertically opposite $\angle$s)

$\angle DAE = \angle ECB$ ($\angle$s in same segment)

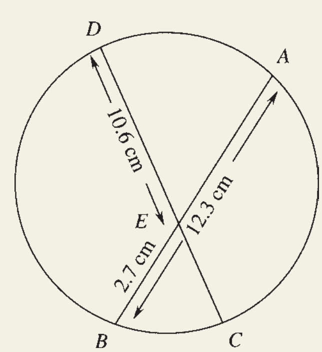
$\angle ADE = \angle EBC$ (similarly)

$\therefore \quad \Delta AED \parallel\!| \Delta CEB$

$$\therefore \quad \frac{AE}{EC} = \frac{DE}{EB}$$

$\therefore AE \cdot EB = DE \cdot EC$

EXAMPLE



Given chord $AB = 12.3$ cm, $EB = 2.7$ cm and $DE = 10.6$ cm, find the length of EC, correct to 1 decimal place.

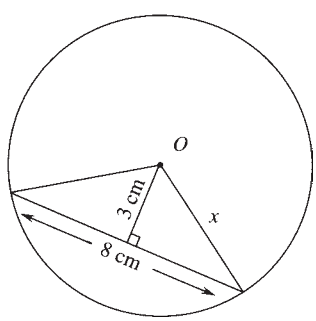
Solution

$$\begin{aligned} AE &= AB - EB \\ &= 12.3 - 2.7 \\ &= 9.6 \\ AE \cdot EB &= DE \cdot EC \\ 9.6 \times 2.7 &= 10.6 \times EC \\ \therefore \quad EC &= \frac{9.6 \times 2.7}{10.6} \\ &= 2.4 \text{ cm} \end{aligned}$$

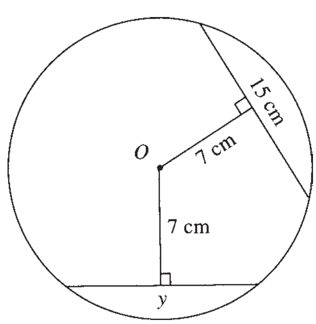
9.2 Exercises

1. Find the values of all pronumerals (O is the centre of each circle).

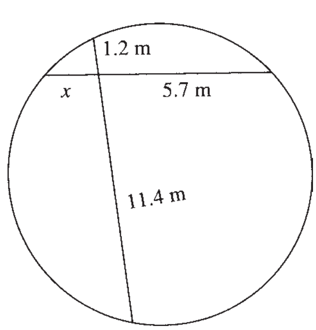
(a)



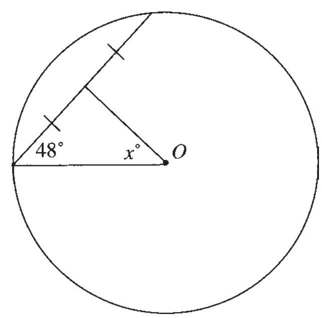
(b)



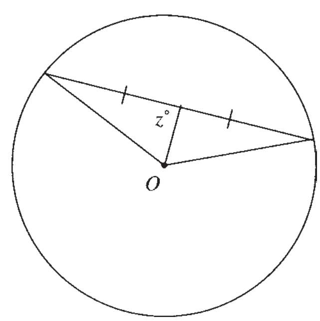
(c)



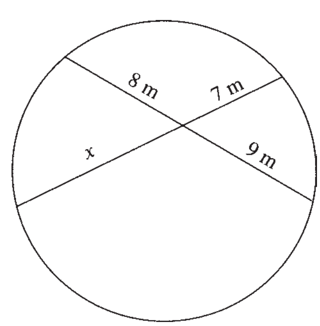
(d)



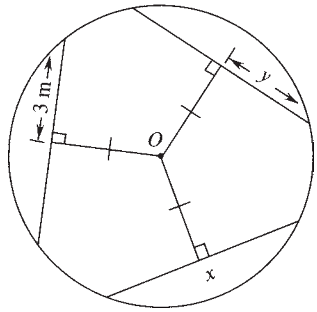
(e)



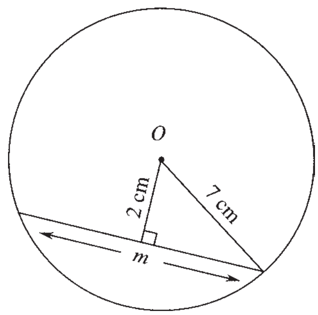
(f)



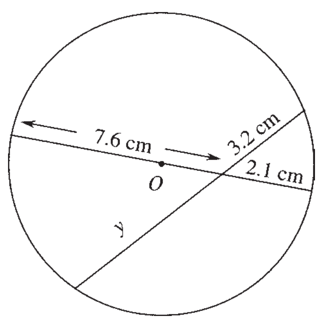
(g)



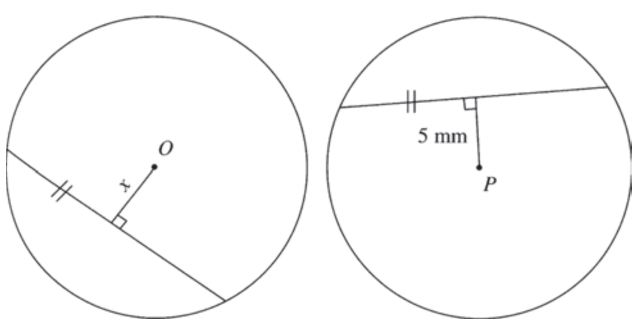
(h)



(i)



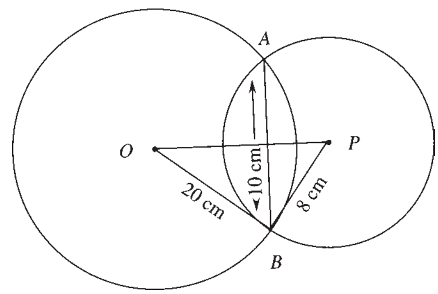
(j)



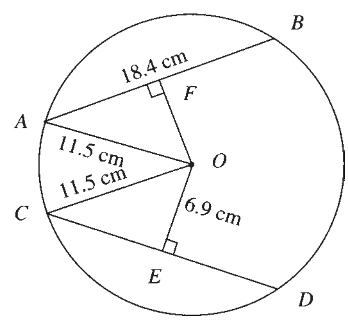
2. Find the exact radius of a circle with a chord that is 8 cm long and 5 cm from the centre.

3. A circle with radius 89 mm has a chord drawn 52 mm from the centre. How long, to the nearest millimetre, is the chord?

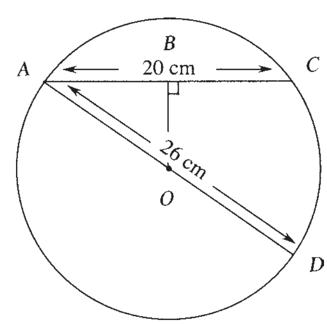
4. O and P are the centres of intersecting circles with radii 20 cm and 8 cm respectively.
If $AB = 10$ cm, find the distance OP, correct to 1 decimal place.



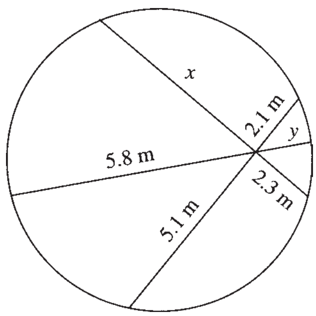
5. Show $AB = CD$.



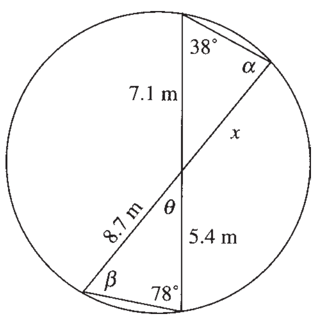
6. $AC = 20$ cm and $AD = 26$ cm. Find OB, correct to 1 decimal place.



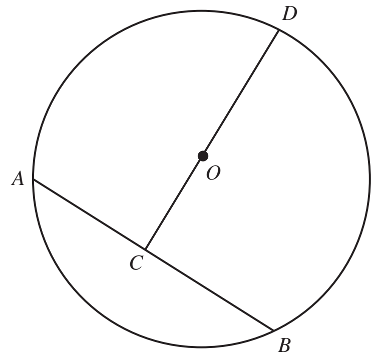
7. Evaluate x and y, correct to 1 decimal place.



8. Find the values of all pronumerals.

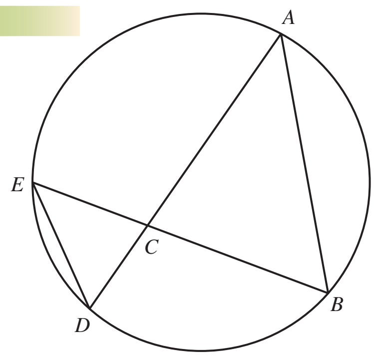


9. A circle with centre O has radius r and chord $AB = x$.



Show that $CD = \dfrac{2r + \sqrt{4r^2 - x^2}}{2}$.

10.



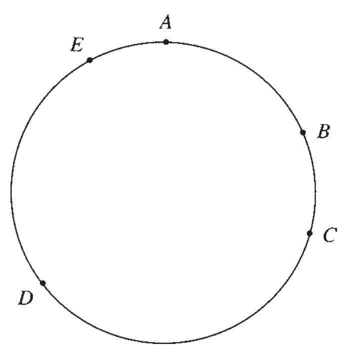
(a) Prove that triangles ABC and CDE are similar.

(b) Show that $AC \cdot CD = BC \cdot CE$.

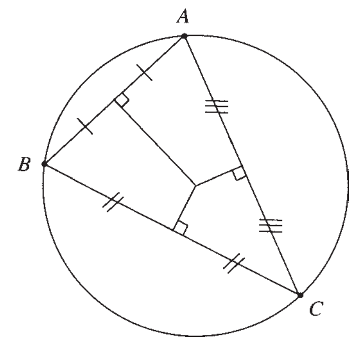
Concyclic Points

EXTENSION

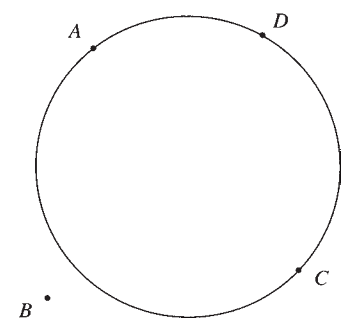
Concyclic points are points that lie on the **circumference** of a circle.



Any 3 **non-collinear** points are **concyclic**. They lie on a unique circle, with centre at the point of intersection of the perpendicular bisectors of the intervals joining these points.

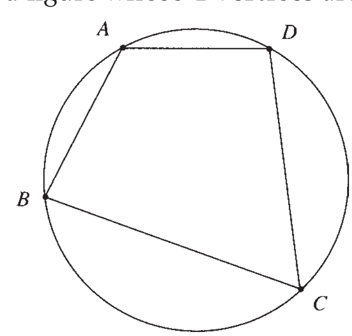


Four or more non-collinear points **may not** necessarily **lie on a circle**.

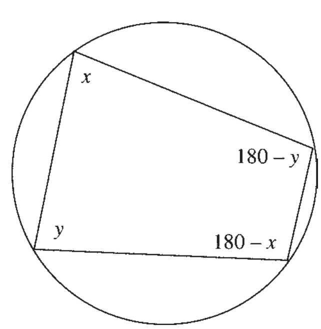


Cyclic quadrilaterals

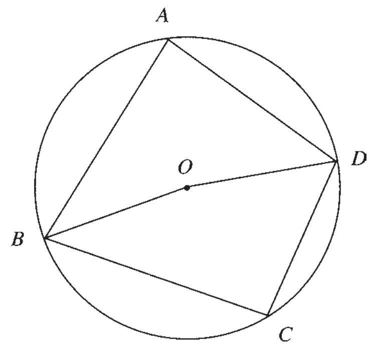
A **cyclic quadrilateral** is a figure whose **4 vertices** are **concyclic points**.



The opposite angles in a cyclic quadrilateral are supplementary.



Proof



Join B and D to O.

$$\begin{aligned} \text{Obtuse } \angle DOB &= 2\angle A && (\angle \text{ at centre is double} \\ \text{Reflex } \angle DOB &= 2\angle C && \quad \angle \text{at circumference}) \\ \text{Obtuse } \angle DOB + \text{reflex } \angle DOB &= 360^\circ && (\angle \text{ of revolution}) \\ \therefore \quad 2\angle A + 2\angle C &= 360^\circ \\ \therefore \quad \angle A + \angle C &= 180^\circ \end{aligned}$$

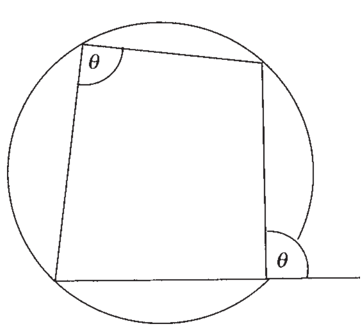
Similarly, it can be proven that $\angle B + \angle D = 180^\circ$ by joining A and C to O.

The converse is also true:

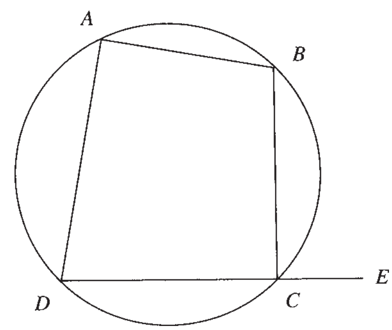
> If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

The property of opposite angles being supplementary in a cyclic quadrilateral can also be used to prove the following property:

> The exterior angle at a vertex of a cyclic quadrilateral is equal to the interior opposite angle.



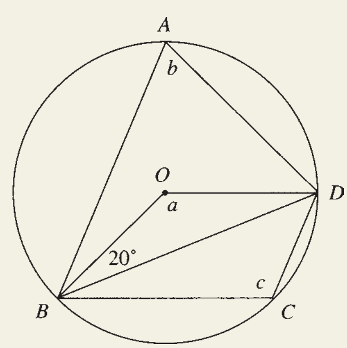
Proof



Let	$\angle A = x$	
Then	$\angle BCD = 180^\circ - x$	(opposite $\angle$s supplementary in cyclic quadrilateral)
$\therefore$	$\angle BCD + \angle BCE = 180^\circ$	($\angle DCE$ straight angle)
$\therefore$	$\angle BCE = 180^\circ - (180^\circ - x)$	
	$= 180^\circ - 180^\circ + x$	
	$= x$	
$\therefore$	$\angle A = \angle BCE$	

EXAMPLE

Evaluate a, b and c.



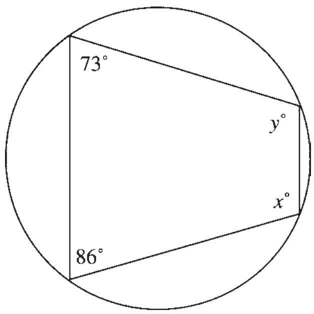
Solution

	$OB = OD$	(equal radii)
$\therefore$	$\angle ODB = 20^\circ$	(base $\angle$s of isosceles Δ equal)
	$a + 20^\circ + 20^\circ = 180^\circ$	($\angle$ sum of Δ)
$\therefore$	$a = 180^\circ - 40^\circ$	
	$= 140^\circ$	
	$b = 70^\circ$	($\angle$ at centre double $\angle$ at circumference)
	$c = 180^\circ - 70^\circ$	(opposite $\angle$s in cyclic quadrilateral)
	$= 110^\circ$	

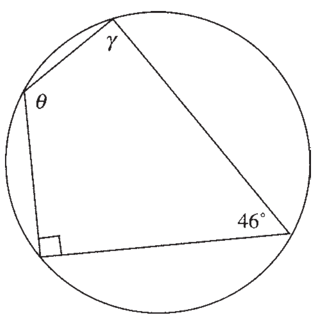
9.3 Exercises

1. Find the values of all pronumerals.

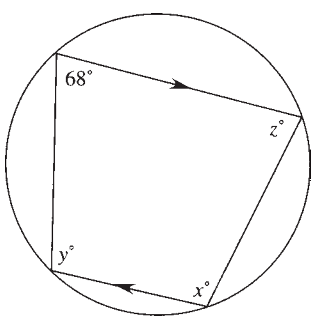
(a)



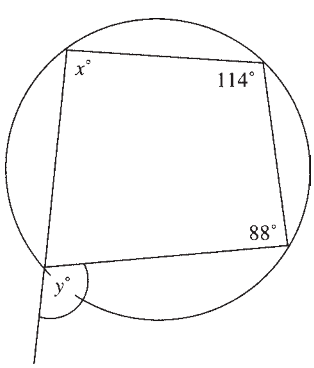
(b)



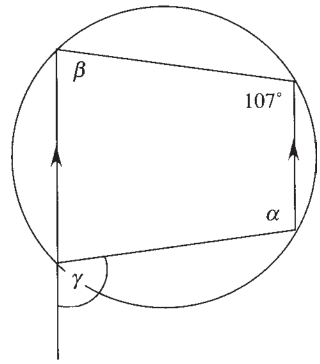
(c)



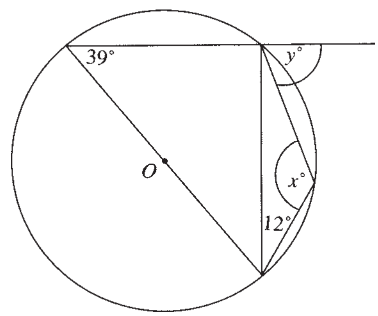
(d)



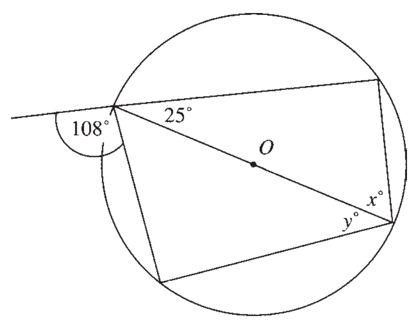
(e)



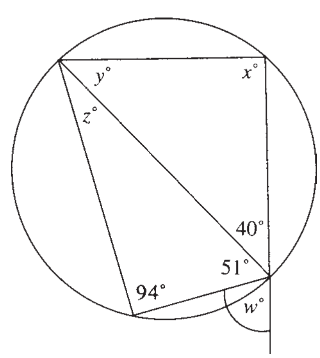
(f)



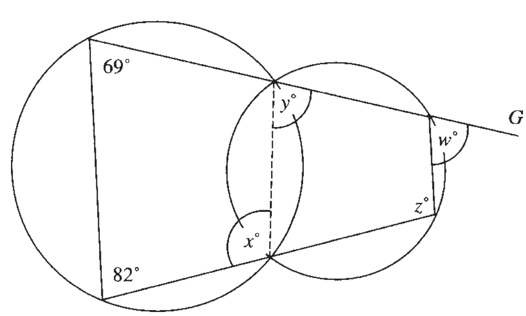
(g)



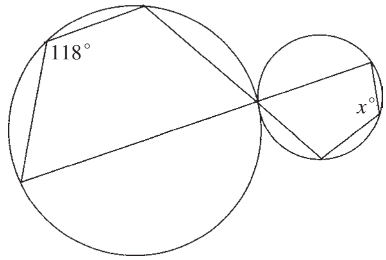
(h)



(i)

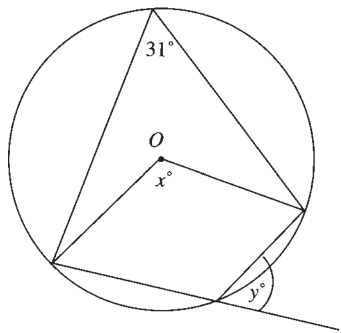


(j)

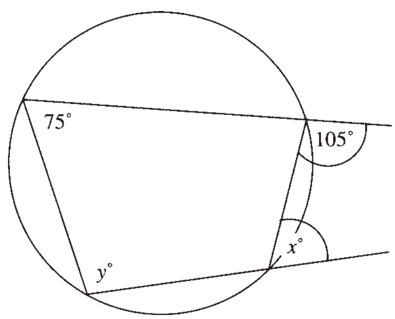


2. Find the values of all pronumerals.

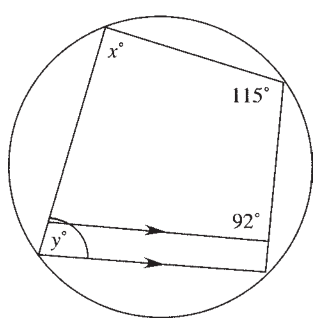
(a)



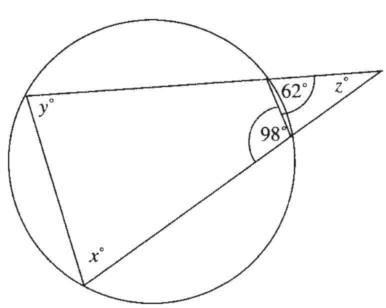
(b)



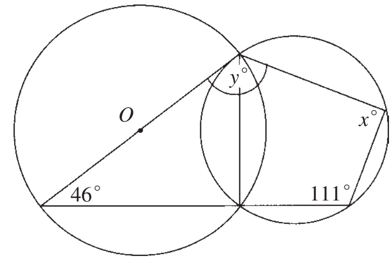
(c)



(d)



(e)



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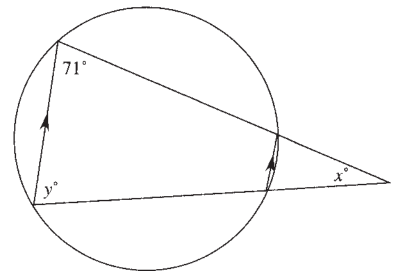
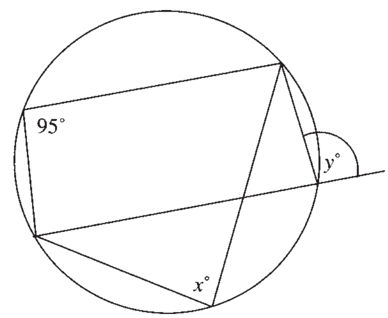
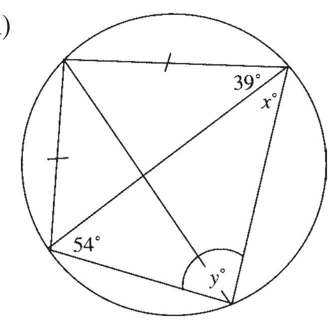


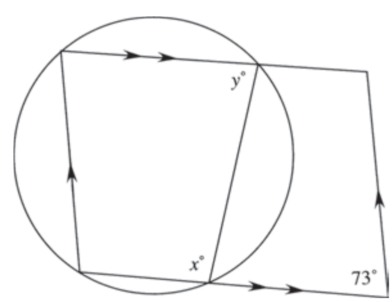
(g)



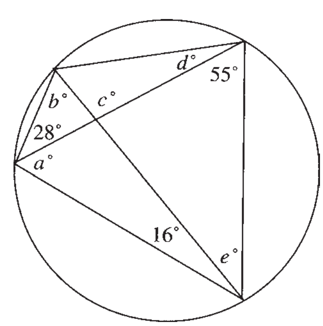
(h)



(i)

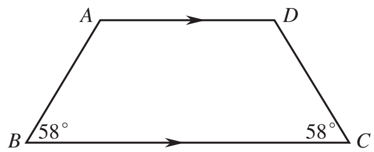


(j)

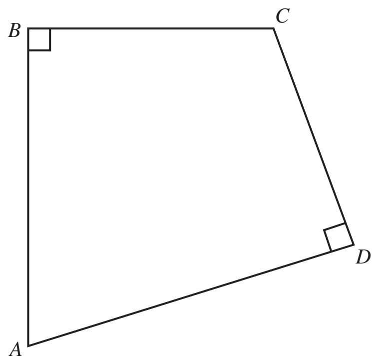


3. Show that $ABCD$ is a cyclic quadrilateral.

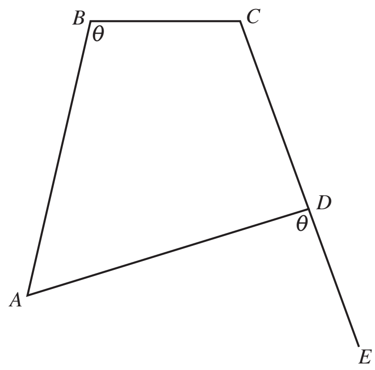
(a)



(b)



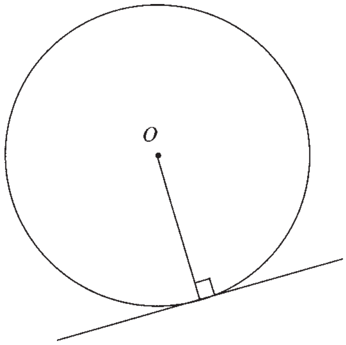
(c)



Tangent Properties

EXTENSION

> The tangent to a circle is perpendicular to the radius drawn from the point of contact.



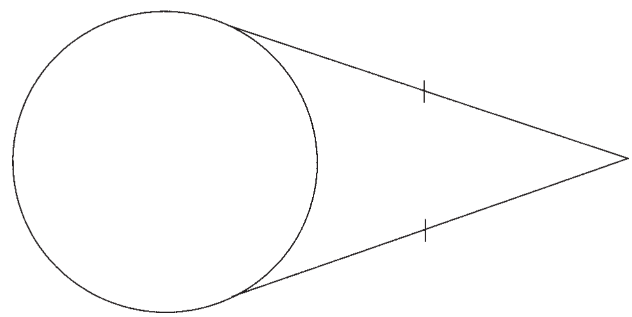
The perpendicular distance is the shortest distance — any other distance would be greater than the radius.

The converse is also true:

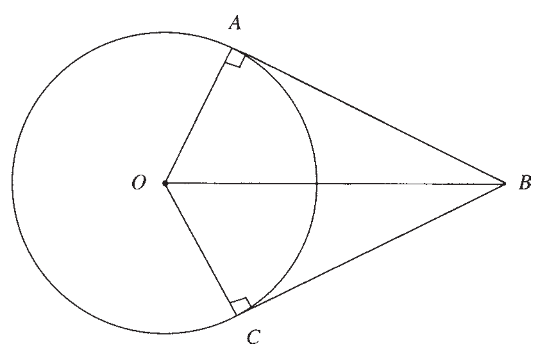
> The line perpendicular to the radius at the point where it meets the circle is a tangent to the circle at that point.

Here is another property of tangents to a circle:

Tangents to a circle from an exterior point are equal.



Proof



Join OB

$\angle A = \angle C = 90^\circ$ (tangent $\perp$ radius)

OB is common

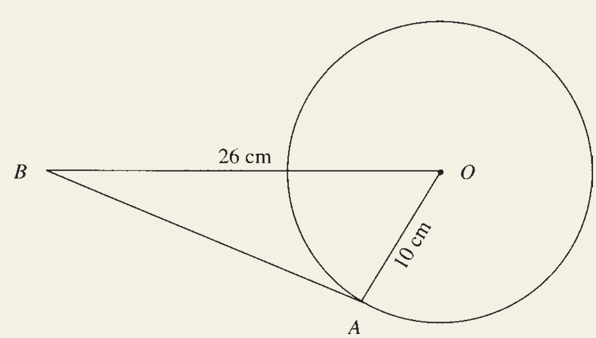
$OA = OC$ (equal radii)

$\therefore$ by RHS, $\Delta OAB \equiv \Delta OCB$

$\therefore$ $AB = CB$ (corresponding sides in congruent Δs)

EXAMPLE

A circle with centre O and radius 10 cm has a tangent AB drawn to it where $OB = 26$ cm. Find the length of AB.



Solution

$OA = 10$	(radius)
$\angle OAB = 90^\circ$	(tangent $\perp$ radius)

$\therefore\ OB^2 = OA^2 + AB^2$

$26^2 = 10^2 + AB^2$

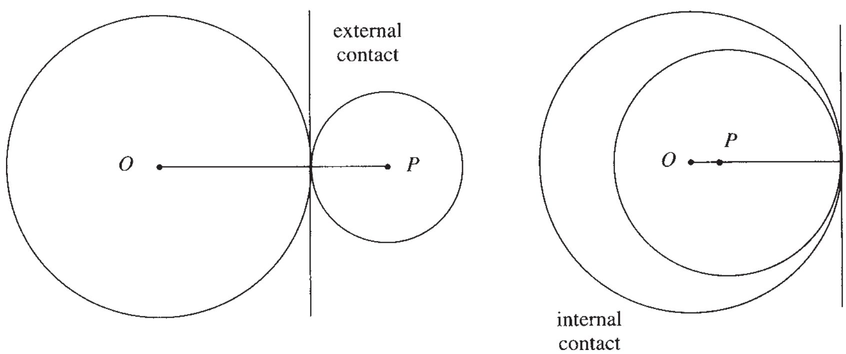
$676 = 100 + AB^2$

$576 = AB^2$

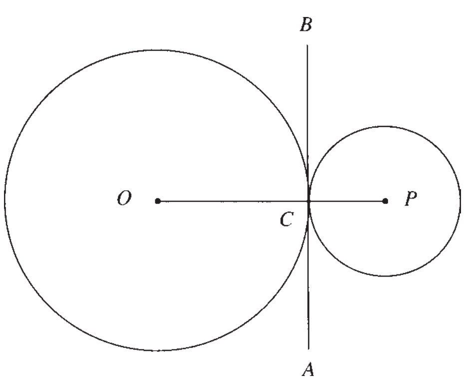
$\therefore\ AB = \sqrt{576}$

$= 24\text{ cm}$

When two circles touch, the line through their centres passes through their point of contact.



Proof



You could also prove this result for when circles touch internally.

AB is a tangent to circle with centre O

$\therefore\ \angle OCB = 90^\circ$ (tangent $\perp$ radius)

AB is a tangent to circle with centre P

$\therefore\ \angle PCB = 90^\circ$ (similarly)

$\angle OCB + \angle PCB = 90^\circ + 90^\circ$

$= 180^\circ$

$\therefore OCP$ is a straight line.

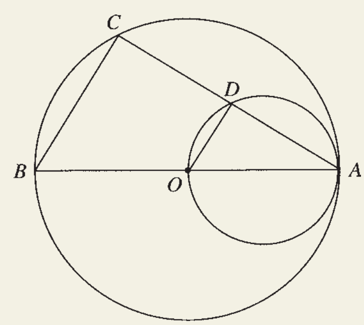
EXAMPLE

Two circles touch at A and the larger circle has centre O. Prove

(a) ΔABC and ΔAOD are similar

(b) $CB \parallel DO$

(c) $BC = 2DO$.



Solution

(a) OA is a diameter of small circle (line through centres passes through point of contact)

$\therefore \angle ODA = 90^\circ$ ($\angle$ in semicircle)

Since AB is a diameter of the larger circle,

$\angle BCA = 90^\circ$ (similarly)

$\therefore \angle BCA = \angle ODA$

$\angle A$ is common

$\therefore \Delta ABC \parallel\!\mid \Delta AOD$

(b) $\angle BCA = \angle ODA$ [from (a)]

These are equal corresponding angles.

$\therefore CB \parallel DO$

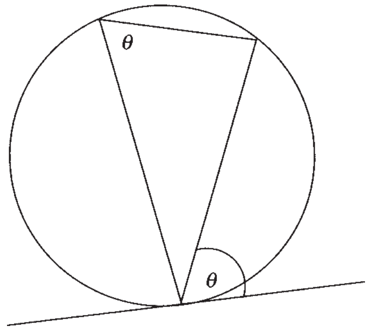
(c) $AB = 2\,OA$ (OA radius)

$\therefore \dfrac{AB}{OA} = 2$

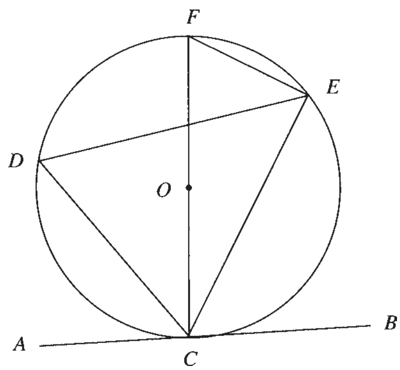
$\therefore \dfrac{AB}{OA} = \dfrac{AC}{AD} = \dfrac{BC}{DO} = 2$ (by similar Δs)

$\therefore BC = 2DO$

The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.



Proof



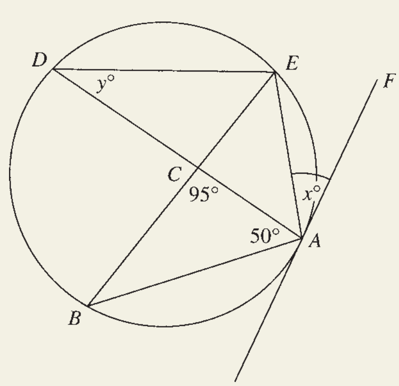
Draw in diameter CF and join EF.

Let $\angle ECB = x$.

$\angle FCB = 90^\circ$	(tangent $\perp$ radius)
$\therefore \angle FCE = 90^\circ - x$	
$\angle FEC = 90^\circ$	($\angle$ in semicircle)
$\therefore \angle EFC = 180^\circ - (90^\circ + 90^\circ - x)$	(angle sum of Δ)
$= x$	
$\angle EFC = \angle EDC$	(angles in same segment)
$\therefore \angle EDC = \angle ECB$	

EXAMPLE

Evaluate x and y.



CONTINUED

Solution

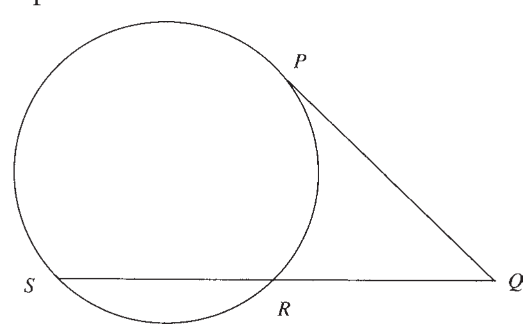
$\angle ABC = 180^\circ - (95^\circ + 50^\circ)$ ($\angle$ sum of Δ)

$= 35^\circ$

$\therefore \quad x = 35^\circ$ ($\angle$ s in alternate segment)

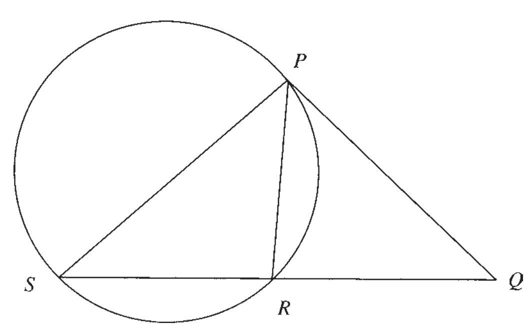
$y = 35^\circ$ (y and $\angle ABC$ in same segment)

The square of the length of the tangent from an external point is equal to the product of the intercepts of the secant passing through this point.



$PQ^2 = QR \cdot QS$ where PQ is a tangent to the circle.

Proof



$\angle QPR = \angle PSR$ (angles in alternate segments)

$\angle Q$ is common

$\therefore PQR \,|||\, \Delta SPQ$

The third pair of angles is equal by angle sum of a triangle.

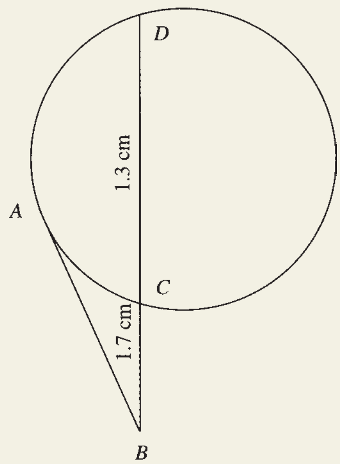
$\therefore \dfrac{PQ}{QS} = \dfrac{PR}{SP} = \dfrac{QR}{PQ}$

$\therefore \dfrac{PQ}{QS} = \dfrac{QR}{PQ}$

$PQ^2 = QR \cdot QS$

EXAMPLE

AB is a tangent to the circle and $CD = 1.3$ cm, $BC = 1.7$ cm. Find the length of AB, correct to 1 decimal place.



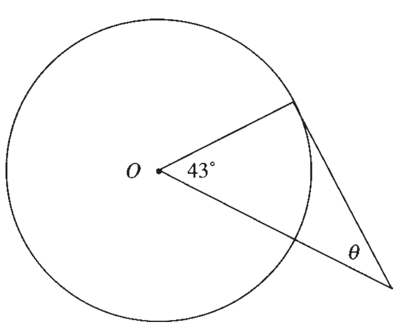
Solution

$$\begin{aligned} BD &= 1.3 + 1.7 \\ &= 3 \\ AB^2 &= BC \cdot BD \\ &= 1.7 \times 3 \\ &= 5.1 \\ AB &= \sqrt{5.1} \\ &= 2.3 \text{ cm correct to 1 decimal place.} \end{aligned}$$

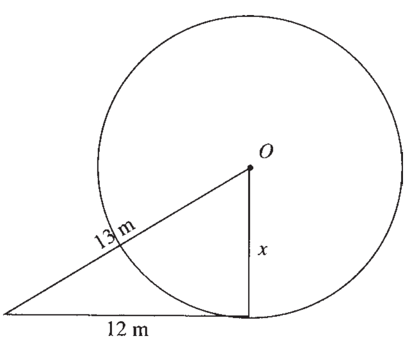
9.4 Exercises

1. Find the values of all pronumerals.

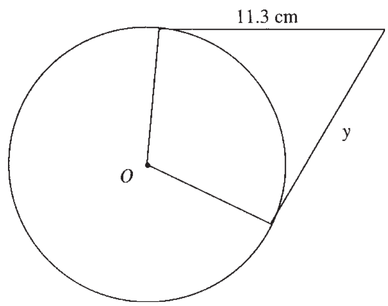
(a)



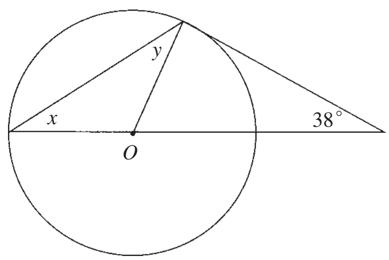
(b)



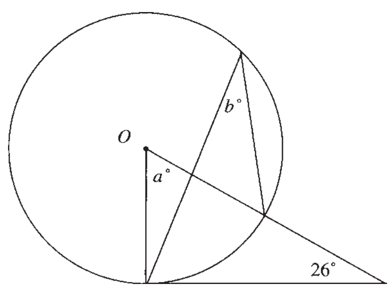
(c)



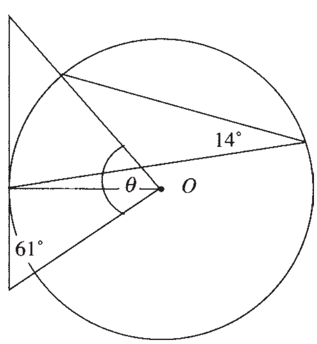
(d)



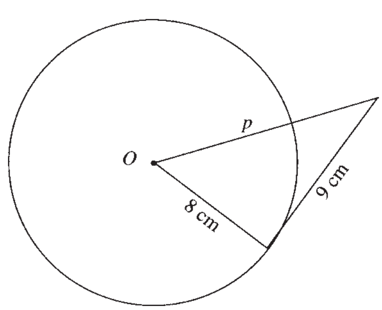
(e)



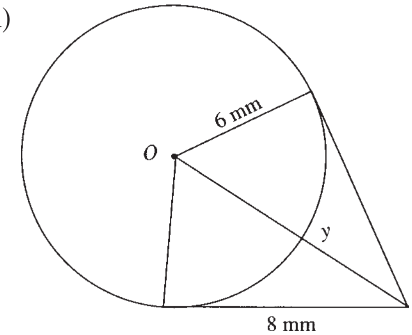
(f)



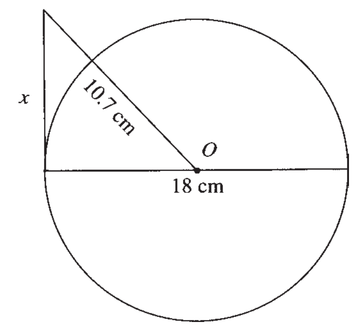
(g)



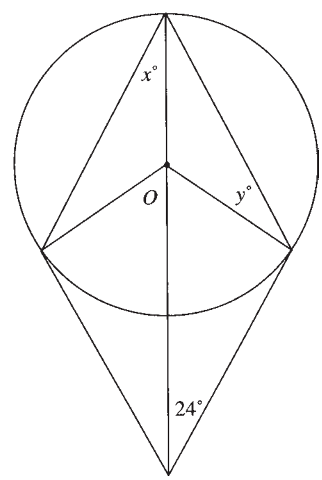
(h)



(i)

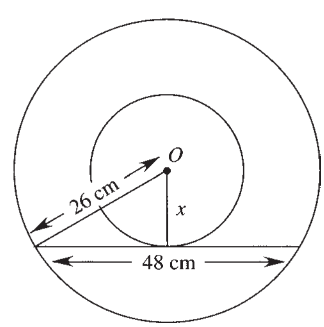


(j)

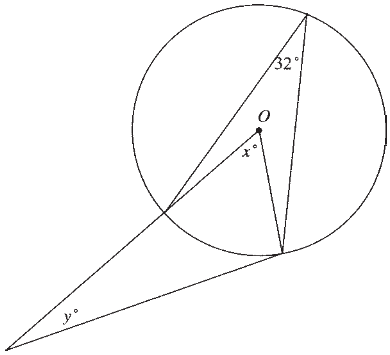


2. Find the values of all pronumerals (all external lines are tangents to the circles).

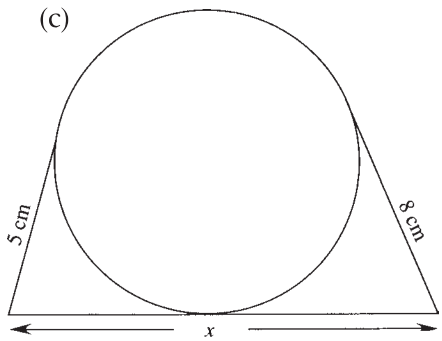
(a)



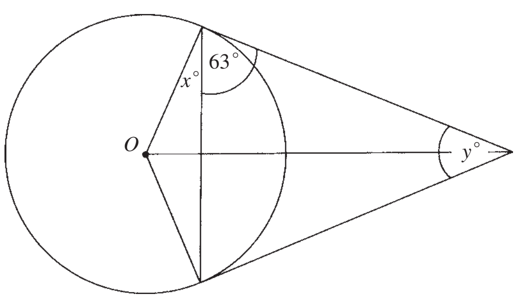
(b)


32°
O
x°
y°

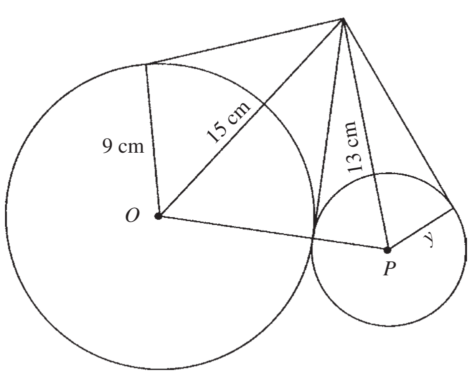
(c)


5 cm
8 cm
x

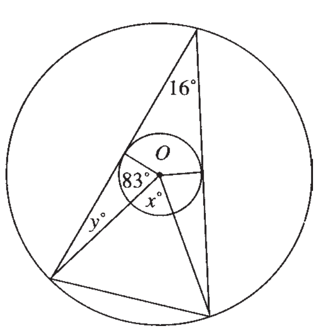
(d)


63°
x°
O
y°

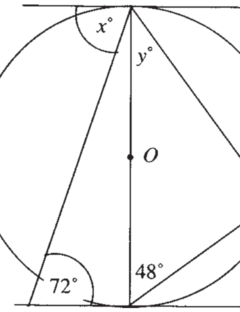
(e)


9 cm
15 cm
13 cm
O
P
y

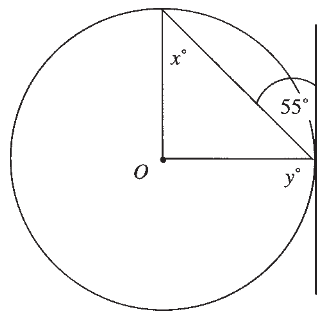
(f)


16°
O
83°
x°
y°

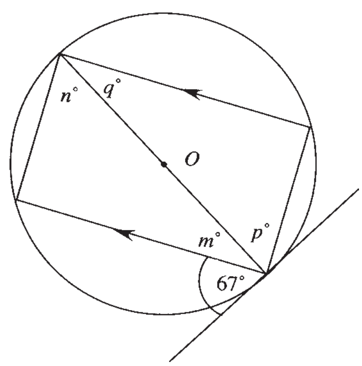
(g)


x°
y°
O
72°
48°

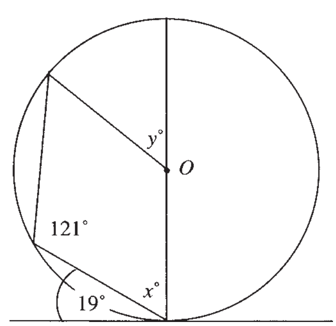
(h)


x°
55°
O
y°

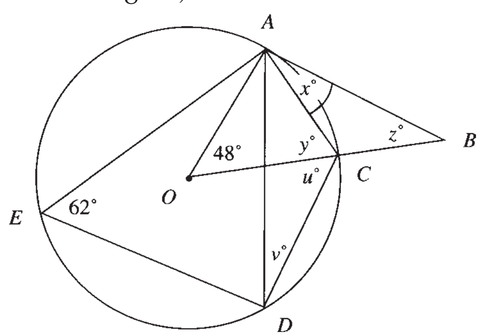
(i)


n°
q°
O
m°
p°
67°

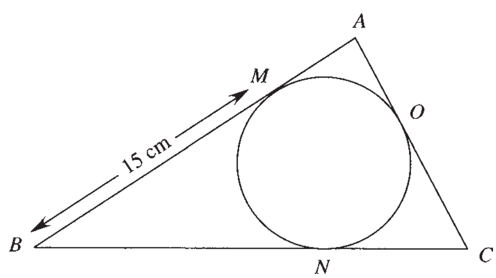
(j)


y°
O
121°
19°
x°

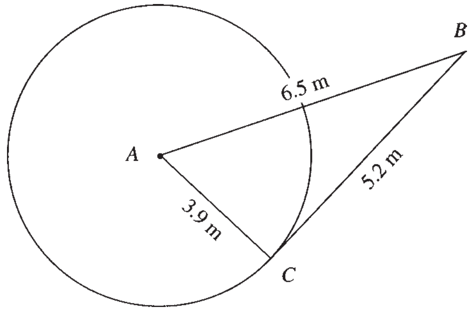
3. Find the values of all pronumerals, giving reasons for each step of your working (O is the centre of circle, AB is a tangent).



4. AB, BC and AC are tangents, with $AB = 24$ cm, $BC = 27$ cm and $BM = 15$ cm. Find the length of AC.

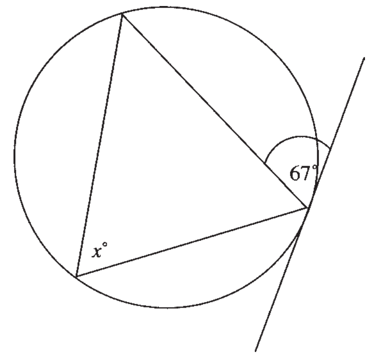


5. $AB = 6.5$ m, $AC = 3.9$ m and $BC = 5.2$ m. Prove A lies on a diameter of the circle, given BC is a tangent to the circle.

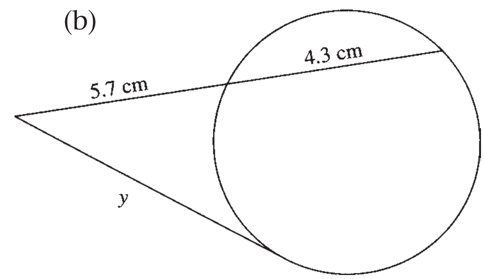


6. Find the values of all pronumerals (O is the centre of each circle; all external lines are tangents).

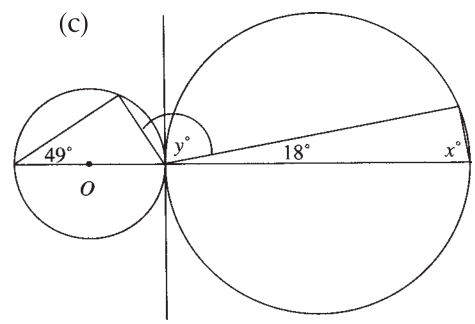
(a)



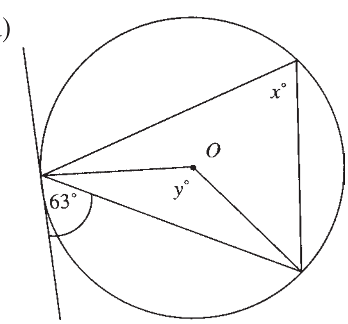
(b)



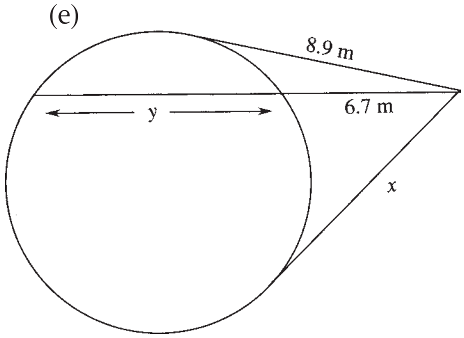
(c)



(d)



(e)



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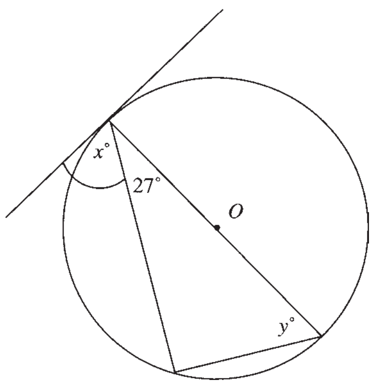
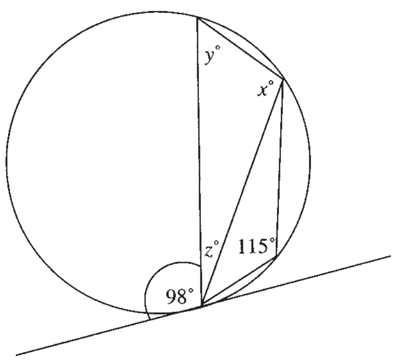
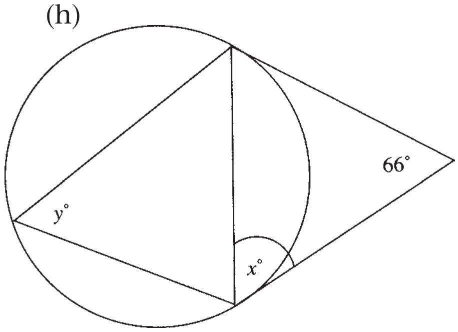


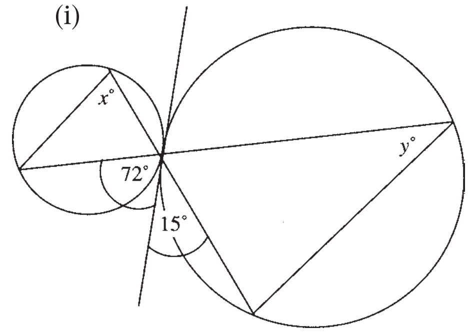
(g)



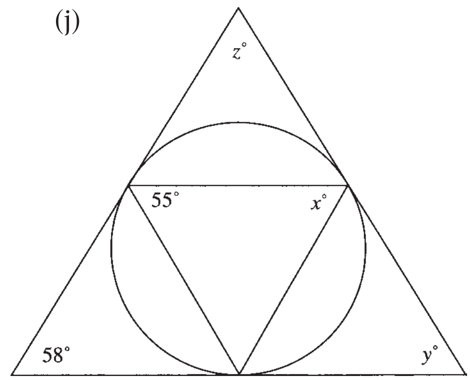
(h)



(i)

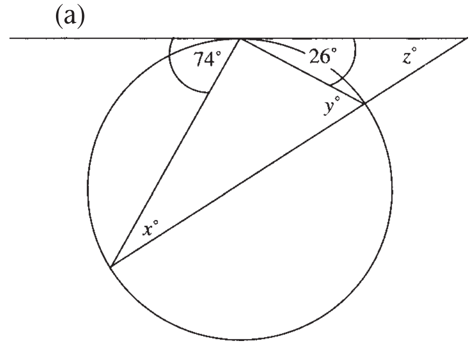


(j)

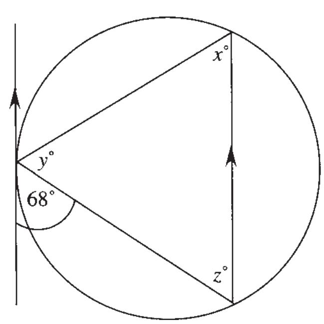


7. Find the values of all pronumerals.

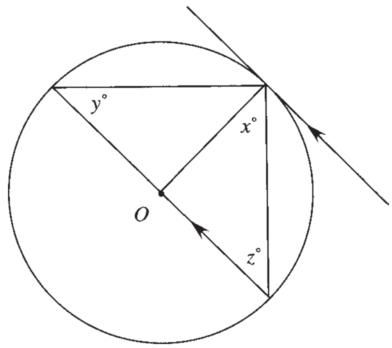
(a)



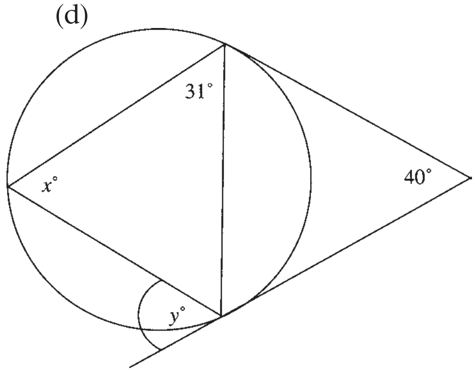
(b)



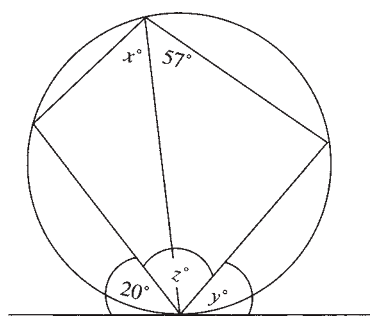
(c)



(d)



(e)



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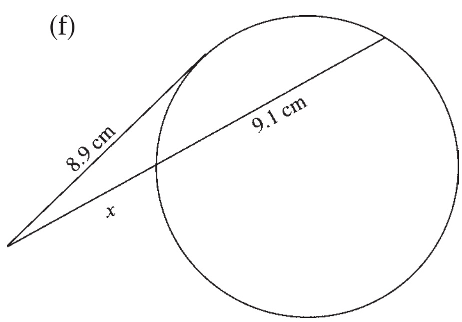

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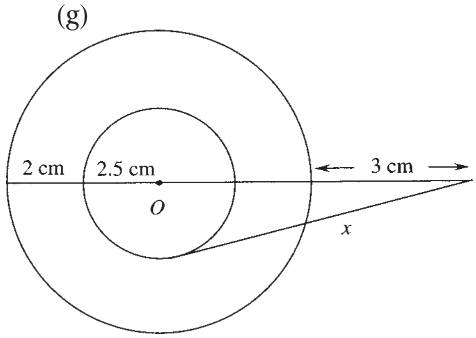
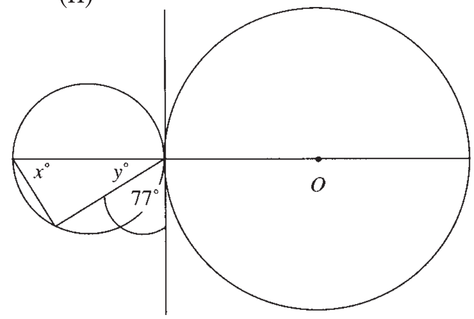
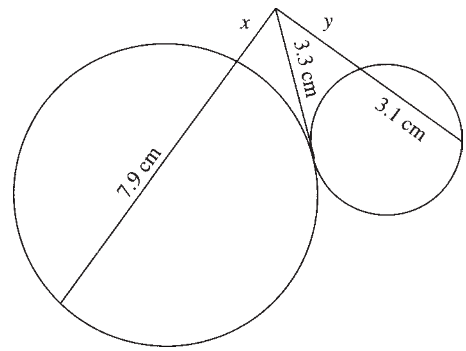


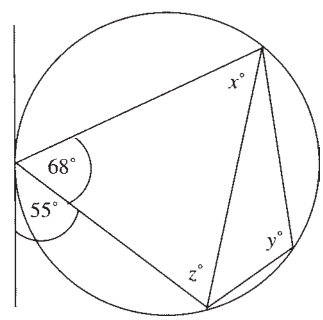
(h)



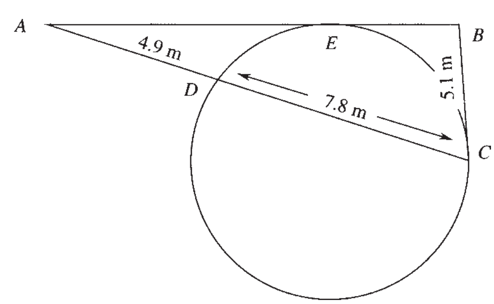
(i)



(j)

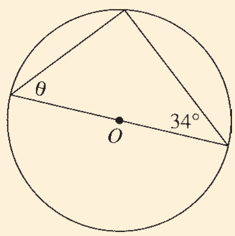


8. Find AB, given $AD = 4.9$ m, $BC = 5.1$ m and $CD = 7.8$ m.

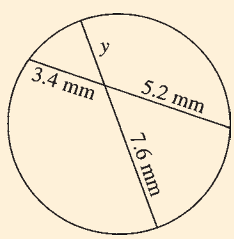


Test Yourself 9

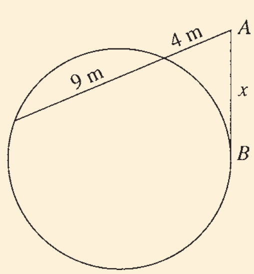
1. O is the centre of the circle. Evaluate θ.



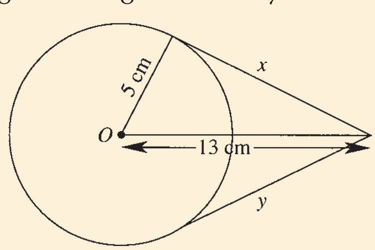
2. Evaluate y to 1 decimal place.



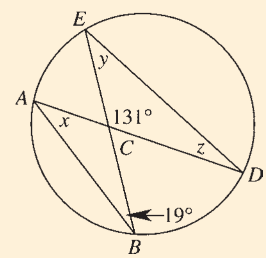
3. AB is a tangent to the circle. Find the value of x to 1 decimal place.



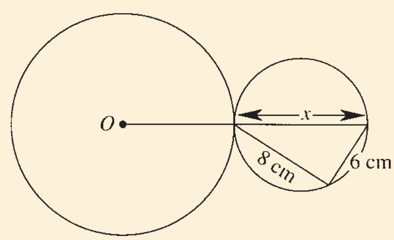
4. O is the centre of the circle. Find the length of tangents x and y.



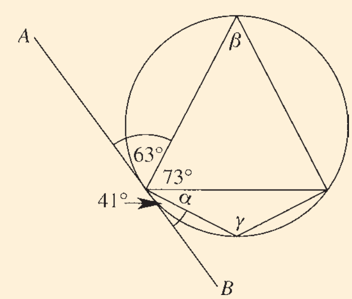
5. Evaluate x, y and z, giving reasons for each step of your working.



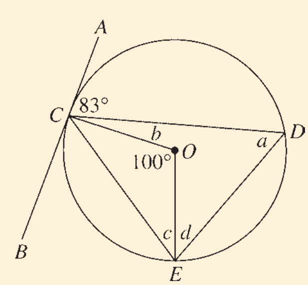
6. O is the centre of the larger circle. Find the value of x.



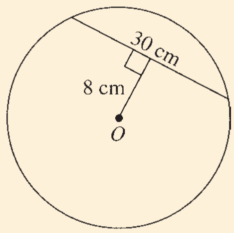
7. AB is a tangent to the circle. Evaluate α, β and γ.



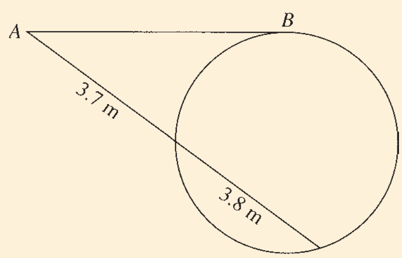
8. O is the centre of the circle, and AB is a tangent. Evaluate a, b, c and d, giving reasons for each step of your working.



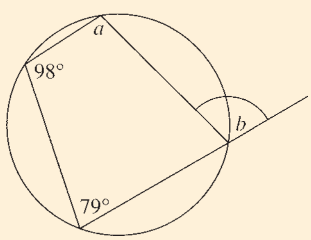
9. Find the length of the radius of the circle. O is the centre.



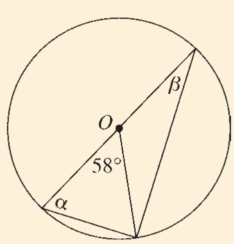
10. Find the length of tangent AB.



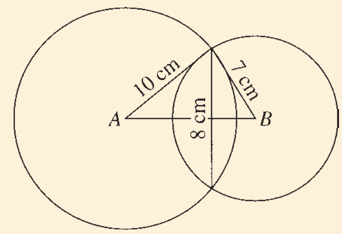
11. Evaluate a and b.



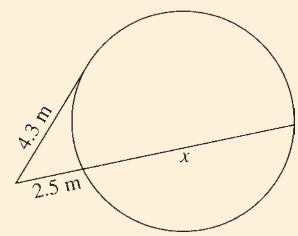
12. O is the centre of the circle. Find the value of α and β.



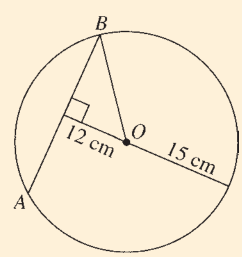
13. Calculate the length of AB to 3 significant figures, given that A and B are the centres of the circles.



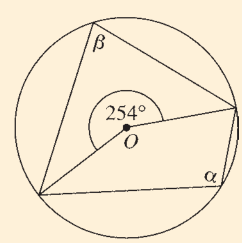
14. Find the value of x to 1 decimal place.



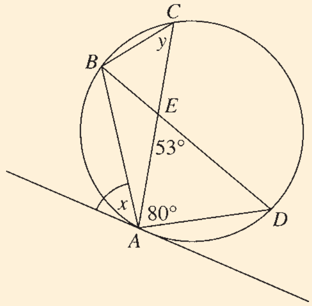
15. Find the length of AB.



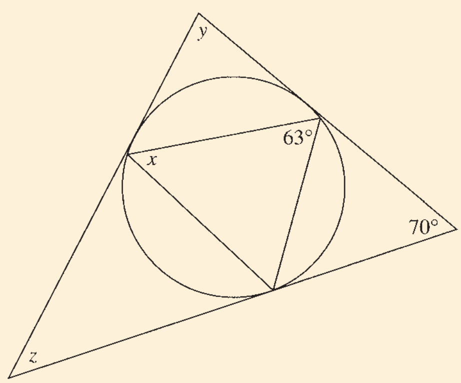
16. Evaluate α and β.



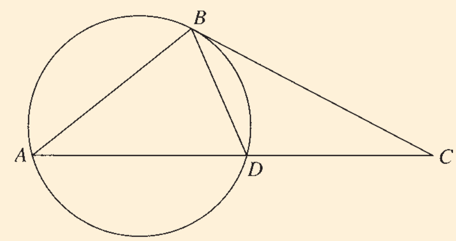
17. Evaluate x and y, giving reasons for your working.



18. Evaluate x, y and z.



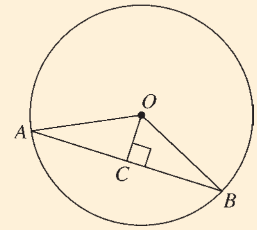
19. Prove that ΔBCD is similar to ΔABC.



20. O is the centre of the circle.

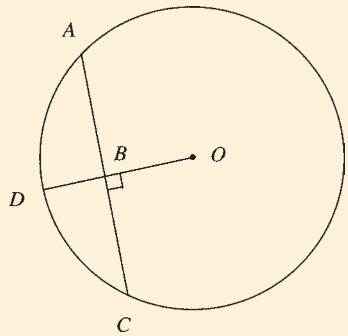
(a) Prove that ΔOAC and ΔOBC are congruent.

(b) Show that OC bisects AB.

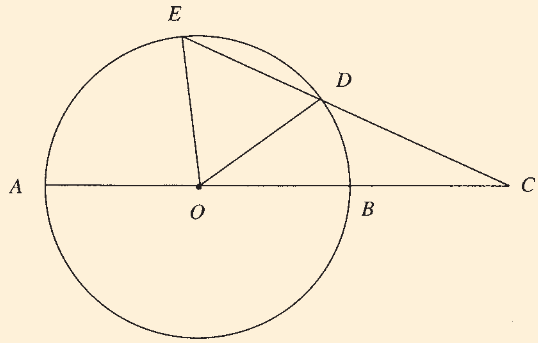


Challenge Exercise 9

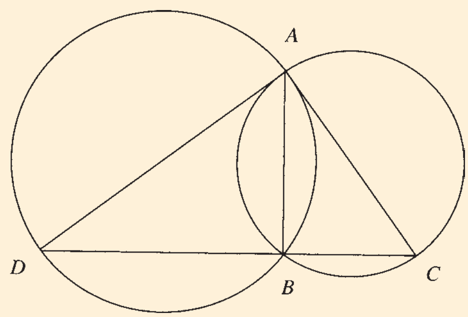
1. Find the length of the radius, to the nearest centimetre, if $AC = 10$ cm and $BD = 3$ cm.



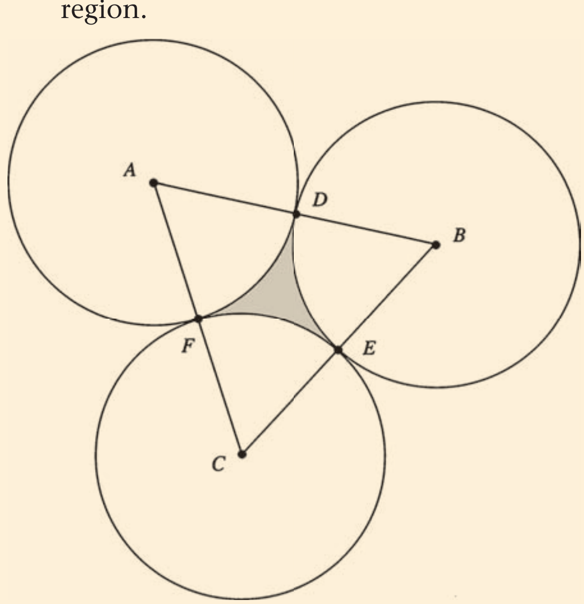
2. In the circle below with centre O, $OD = DC$. Prove $\angle AOE = 3\angle DCB$.



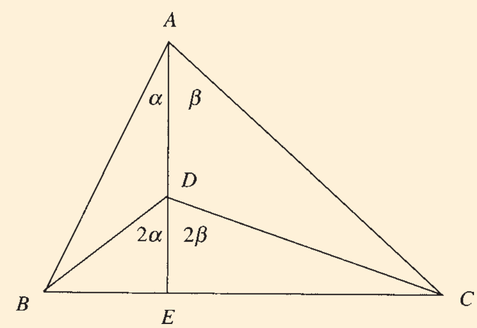
3. Two circles meet at points A and B. A tangent to each circle is drawn from A to meet the circles at D and C. Prove $\angle DAC = 90^\circ$.



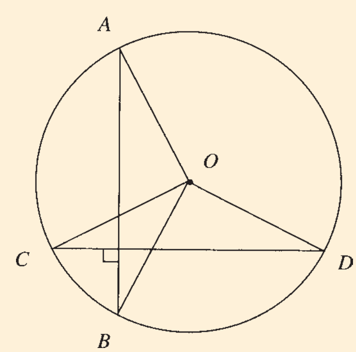
4. Three equal circles touch each other, as in the figure.
(a) Prove that the triangle with vertices the centres of the circles is equilateral.
(b) Find the perimeter of the curved figure DEF in terms of the radius r of the circles.
(c) Find the exact area of the shaded region.



5. The triangles below have $\angle BDE = 2\angle BAD$ and $\angle CDE = 2\angle CAD$. Prove that a circle can be drawn through A, B and C with centre D.

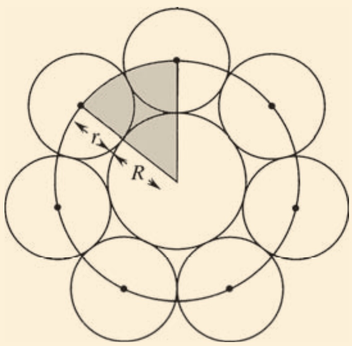


6. Two chords AB and CD intersect at 90° Prove, for obtuse $\angle AOD, \angle AOD + \angle COB = 180^\circ$ where O is the centre of the circle.



7. Prove that any kite $ABCD$ with $\angle ADC = \angle ABC = 90^\circ$ is a cyclic quadrilateral with diameter AC.

8. A large circle with radius R is surrounded by 7 smaller circles with radius r. A circle is drawn through the centres of the smaller circles. If $R = \frac{3r}{2}$, find the shaded area in terms of r.



9. Prove that if an interval subtends equal angles at two points on the same side of it, then the endpoints of the interval and the two points are concyclic.

10. Prove that if both pairs of opposite angles in a quadrilateral are supplementary, then the quadrilateral is cyclic.